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A novel granular computing model based on three-way decision

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ABSTRACT

Granular computing and three-way decision are two very important methods in the field of knowledge discovery and data mining. In this paper, based on the idea of three-way decision, all attributes in the information table first are divided into three disjoint parts named indispensable attributes, rejected attributes and neutral attributes, respectively. According to the three parts of attributes, many basic and important information granules and granular structures can be induced from the information table. Then a novel granular computing model is proposed by the description operator. On the one hand, many mathematical properties related to the model proposed in this paper are systematically discussed. On the other hand, we make a preliminary and meaningful attempt to deal with network security by using this model. In addition, in order to apply the model more conveniently, two algorithms for computing description set, description degree, attribute reduction and reduction degree are developed. Finally, through numerical experiments, the validity of the algorithms and the related factors that affect the effectiveness of the algorithms are discussed in detail.

1. Introduction

Data has penetrated into every industry and field in today's society and become an important production factor. Now we often need to face and deal with massive data (research objects), and these data have a variety of characteristics (attributes) and characteristic values (attribute values). We can usually illustrate these data information through an information table. Due to limited cognitive ability, the complex data on a large scale can be divided into lots of simple blocks according to the characteristics of the attributes in an information table. These blocks are usually treated as information granules. Then we can analyze the data based on these information granules and extract useful knowledge. Therefore, this idea of granulating and processing the complex data is called granular computing method. In recent years, granular computing has become a popular research branch in the fields of knowledge discovery and data analysis [1–4].

Since Zadeh published a paper entitled "fuzzy sets and information granularity" in 1979 [5], researchers have been interested in the idea of information granulation. According to the different actual requirements and the data characteristics,

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scholars adopted different methods and rules to granulate the data. Therefore, many granular computing models were developed based on various information granules. For example, Pawlak proposed a rough set model in 1982 [6]. Its essential idea is to define two exact sets (upper and lower approximation sets) by using a partition on the universe and describe a set with fuzzy boundary. In 1985, Hobbs discussed the decomposition and merging of granules, and how to get granules with different sizes, and proposed a model to generate granules with different sizes [7]. Zhang put forward the quotient space model when they studied the problem solving [8]. First, different quotient spaces can be constructed for the same problem. Then we can get various solutions from different angles and levels. Finally, based on these solutions, the solution of the original problem can be accurately described. Lin discussed the granular computing model in binary relation, and explored the granular structure, granular description and granular application [9–11]. On the basis of Lin's work, Yao proposed the granular computing model based on neighborhood system [12–14]. This model leaded to solve the problem of consistent classification by using the lattice composed of all partitions and provided a new method and perspective for knowledge mining. Based on probability theory and fuzzy mathematical theory, Li introduced the cloud model which can realize the mutual conversion between qualitative and quantitative [15]. Zheng et al. put forward a granular computing model based on tolerance relation [16,17]. Of course, the granular computing theory can be also combined with other related theories and methods to obtain many effective methods for processing data [18–26]. To sum up, we can see that scholars defined information granules by different methods and then built corresponding granular computing models to solve various theoretical and practical problems.

As a wonderful number, "3" plays an important role in people's daily life and work. Three-way decisions are ternary or ternary thinking, which can also be understood as a granular computing model based on three granules. By adding the "uncommitted" option to the traditional "accept" and "reject" options, Yao proposed a three-way decision model by dividing the research objects into three disjoint parts, which can effectively avoid the loss caused by the false acceptance or rejection under the uncertainty of object cognition, and improve the accuracy of decision [27–30]. It has been proved that the three-way decision theory is a very effective method to deal with data, and has made a lot of achievements in many fields, such as three-way concept learning [31,32], three-way concept analysis [33–35], three-way clustering [36–39], three-way classification [40,41], three-way attribute reduction [42–44], fuzzy three-way approximations [45], and others [46–49].

Meanwhile, conflict analysis and resolution plays an important role in data mining, business, governmental, political and lawsuit disputes, labor-management negotiations, military operations and others [50,51]. For example, by considering an example of the Middle East conflict, Pawlak proposed a new conflict analysis model by assigning attribute values to + 1, - 1 and 0, where + 1, - 1 and 0 respectively represent three attitudes of the agent to support, oppose and neutral something [52]. Recently, combined with the idea of three-way decision, the three-way conflict analysis models have been established and widely studied [53–55].

According to the discussion in the preceding two paragraphs, we have further findings. In an information table, research objects often have many attributes. Because of the different emphases of our research and concern, the status and function of these attributes will be different. One often find that some attributes are necessary; some attributes are unnecessary; and the rest are dispensable. Based on the idea of three-way decision, we can divide all attributes into three disjoint parts. According to these three disjoint parts, we will define an information granule. With different problems and concerns, the attributes of these three parts could be changed, and the information granules will be various. In this way, we can gather all the information granules together and develop a new granular structure. Next, we can introduce a novel granular computing model which has clear meanings and good mathematical properties. In addition, the development of network technology not only brings convenience to people's life, but also produces a lot of security risks. With the frequent occurrence of network security events in recent years, people have paid more attention to network security issues than before. More and more scholars have begun to study various network security issues [56,57]. So, we also study the representation and reduction of a class of network security by using the granular computing model introduced in this paper. Finally, we design two algorithms for computing description set and reduction, and test the effectiveness of the algorithms through numerical experiments.

The rest of this paper is organized as follows. In Section 2, based on the three-way idea, all attributes in an information table are divided into three disjoint parts, and a new kind of information granule is constructed. Then the concrete meanings of this kind of information granule are explained. At the same time, a more general granular structure is given, which is a generalization of those granular structures used in rough set theory. In Section 3, according to the granular structure proposed in this paper, a novel granular computing model is explored. The concrete meanings and computing properties of this model are studied deeply. In Section 4, as a practical application of the model, one can find that the model has a very good performance in dealing with the information network security. In Section 5, in order to use the model to solve practical problems, two algorithms are designed to compute description set, description degree, attribute reduction and reduction degree, respectively. In Section 6, experimental analysis shows that the two algorithms have ideal time consumption. And the related factors that affect the effectiveness of the algorithms are discussed in detail. Section 7 gives a brief review and summary of this paper and further introduces the follow-up research works.

Table 1
An information table.

OB	a_1	a_2	a_3	a_4	a_5	a_6
o_1	0	1	1	1	0	0
o_2	1	0	0	1	0	0
o_3	0	0	1	0	1	0
o_4	0	1	0	0	0	0
o_5	1	0	1	1	0	1
o_6	0	0	0	1	0	0
o_7	1	0	0	1	0	1
o_8	0	1	0	1	0	0

2. The information granules and granular structure based on three-way decision

An information table is a table, rows of which are labeled by objects, columns are labeled by attributes and entries of the table are values of attributes, which are uniquely assigned to each object and each attribute. Usually, an information table can be defined by a tuple as follows [58].

$$I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\}),$$

where the universe $OB = \{o_1, o_2, \cdots, o_n\}$ is a nonempty finite set; $AT = \{a_1, a_2, \cdots, a_m\}$ is a nonempty attribute set; $V = \cup_{a\in A} V_a$, V_a is the domain of attribute a; $f_a : OB \to V_a$ is an information function. We use $f_a(o)$ to denote the value of object o on attribute a. In this paper, we suppose that $V_a = \{0, 1\}$, where $f_a(o) = 1$ means that o has the attribute a; $f_a(o) = 0$ means that o does not have the attribute a, $i = 1, 2, \cdots, m$. Meanwhile, for any $o \in OB$, there exists $a \in AT$ such that $f_a(o) = 1$.

According to the information table, many granular structures and granular computing models have been constructed. Especially, using the idea of three-way decision, objects can be divided into three categories, and corresponding granular computing model can be constructed to analyze and mine data in the information table. At the same time, we can also let the attribute values in the information table take three numbers, and build a conflict analysis model to solve the relevant problems in the game theory.

Similarly, when we analyze the information in the information table and solve some data problems, we will find that the attributes of data can be divided into three disjoint parts. For research problems, the attributes of the first part must appear, the attributes of the second part must not appear, and whether the attributes of the last part appear or not has no influence on the research problems. For instance, in the selection of scholarships, all candidates must have excellent examination results and social practice experience, have no cheating record and can not be international students. However, there is no requirement for the candidates' sports ability and health.

Based on the above description, an information table shown in Table 1 can be developed, where we take the students as the research objects and regard international student, excellent examination results, excellent athletic ability, social practice, history of major diseases and cheating records as the attributes.

Example 2.1. Let $I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$ be an information table, where $OB = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8\}$, $AT = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. a_1, a_2, a_3, a_4, a_5 and a_6 respectively represent international student, excellent examination results, excellent athletic ability, social practice, history of major diseases and cheating records. In addition, $f_{a_i}(o_j) = 1$ means that o_j has the attribute a_i; $f_{a_i}(o_j) = 0$ means that o_j does not have the attribute a_i, $i = 1, 2, 3, 4, 5, 6$; $j = 1, 2, 3, 4, 5, 6, 7, 8$. More details can be shown in Table 1.

According to Table 1, here are some basic information granules, which are very important and will be constantly used in the following sections. For example, two basic information granules with respect to attribute a_1 are

$$g_{a_1} = \{o_2, o_5, o_7\} \text{ and } \overline{g_{a_1}} = \{o_1, o_3, o_4, o_6, o_8\},$$

where g_{a_1} and $\overline{g_{a_1}}$ are two sets of all international students and all non international students, respectively. Similarly, for any other attribute, we can also get two basic information granules. In this way, we can obtain twelve basic information granules, which are $g_{a_1}, \overline{g_{a_1}}, g_{a_2}, \overline{g_{a_2}}, \cdots, g_{a_6}, \overline{g_{a_6}}$.

According to these twelve basic granules, all the students who meet the requirements of applying for scholarship can be computed as follows:

$$\begin{aligned}
&(g_{a_2} \cap g_{a_4}) \cap (\overline{g_{a_1}} \cap \overline{g_{a_6}}) \\
=&(\{o_1, o_4, o_8\} \cap \{o_1, o_2, o_5, o_6, o_7, o_8\}) \\
&\cap (\{o_1, o_3, o_4, o_6, o_8\} \cap \{o_1, o_2, o_3, o_4, o_6, o_8\}) \\
=&\{o_1, o_8\}
\end{aligned}$$

Let's consider another case, some students need to be selected to participate in the sports meeting. The specific requirement is that all candidates have excellent athletic ability and good health. Similarly, all students who meet the requirements can be presented as follows:

$$\begin{aligned} & g_{a_3} \cap \overline{g_{a_5}} \\ =& \{o_1, o_3, o_5\} \cap \{o_1, o_2, o_4, o_5, o_6, o_7, o_8\} \\ =& \{o_1, o_5\} \end{aligned}$$

Inspired by Example 2.1, for an information table $I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$, and any $a \in A$, g_a is the set of all objects with attribute a; $\overline{g_a}$ is the set of all objects without attribute a. For $A_s, A_t \subseteq AT$, where $\emptyset \subseteq A_s, A_t \subseteq AT$, and $A_s \cap A_t = \emptyset$, then the attribute set AT can be divided into three disjoint parts: A_s, A_t, and $A_r = AT/(A_s \cup A_t)$. The attributes in A_s, A_t and A_r are called indispensable attributes, rejected attributes and neutral attributes, respectively. Then all objects that have attributes in A_s, don't have attributes in A_t, and have no restrictions on attributes in A_r can be proposed as follows:

$$\begin{aligned} g^I_{A_s,A_t} &= (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}}) \cap (\cap_{a_k \in A_r} (g_{a_k} \cup \overline{g_{a_k}})) \\ &= (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}}) \end{aligned}$$

Generally speaking, $g^I_{A_s,A_t}$ is determined by three attribute sets A_s, A_t and A_r. At the same time, according to the structure of $g^I_{A_s,A_t}$, two special cases about $g^I_{A_s,A_t}$ are shown as follows.

Special case 1: AT can be divided into two disjoint parts:

(a) If $A_r = \emptyset$, $A_s \neq \emptyset$, $A_t \neq \emptyset$, then

$$g^I_{A_s,A_t} = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in AT/A_s} \overline{g_{a_j}}).$$

Information granule $g^I_{A_s,A_t}$ is determined by A_s and A_t, which is the set of all objects with only attributes in A_s. In other words, $g^I_{A_s,A_t}$ is the set of all objects without only attributes in A_t;

(b) If $A_t = \emptyset$, $A_s \neq \emptyset$, $A_r \neq \emptyset$, then

$$g^I_{A_s,A_t} = g^I_{A_s,\emptyset} = \cap_{a_i \in A_s} g_{a_i}.$$

Information granule $g^I_{A_s,A_t}$ is developed based on A_s and A_r, which is the set of all objects with attributes in A_s;

(c) If $A_s = \emptyset$, $A_t \neq \emptyset$, $A_r \neq \emptyset$, then

$$g^I_{A_s,A_t} = g^I_{\emptyset,A_t} = \cap_{a_j \in A_t} \overline{g_{a_j}}.$$

Information granule $g^I_{A_s,A_t}$ is constructed by A_t and A_r, which is the set of all objects without attributes in A_t.

Special case 2: AT can be divided into only one part:

(d) If $A_t = A_r = \emptyset$, $A_s = AT$, then

$$g^I_{A_s,A_t} = g^I_{AT,\emptyset} = \cap_{a_i \in AT} g_{a_i}.$$

Information granule $g^I_{A_s,A_t}$ is only related to A_s, which is the set of all objects with attributes in AT;

(e) If $A_s = A_r = \emptyset$, $A_t = AT$, then

$$g^I_{A_s,A_t} = g^I_{\emptyset,AT} = \cap_{a_j \in AT} \overline{g_{a_j}}.$$

Information granule $g^I_{A_s,A_t}$ is only determined by A_t, which is the set of all objects without attributes in AT.

In addition, based on the structure of $g^I_{A_s,A_t}$, the followed properties with respect to set union and intersection can be shown as follows.

Proposition 2.1. *Suppose that $I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$ is an information table, then the following results hold.*

(1) $g^I_{A_{s_1},A_{t_1}} \cap g^I_{A_{s_2},A_{t_2}} = g^I_{A_{s_1} \cap A_{s_2}, A_{t_1} \cap A_{t_2}}$;

(2) $g^I_{A_{s_1},A_{t_1}} \cup g^I_{A_{s_2},A_{t_2}} \subseteq g^I_{A_{s_1} \cup A_{s_2}, A_{t_1} \cup A_{t_2}}$.

Proof. (1) $g^I_{A_{s_1},A_{t_1}} \cap g^I_{A_{s_2},A_{t_2}} = (\cap_{a_i \in A_{s_1}} g_{a_i}) \cap (\cap_{a_j \in A_{t_1}} \overline{g_{a_j}}) \cap (\cap_{a_i \in A_{s_2}} g_{a_i}) \cap (\cap_{a_j \in A_{t_2}} \overline{g_{a_j}}) = ((\cap_{a_i \in A_{s_1}} g_{a_i}) \cap (\cap_{a_i \in A_{s_2}} g_{a_i})) \cap ((\cap_{a_j \in A_{t_1}} \overline{g_{a_j}}) \cap (\cap_{a_j \in A_{t_2}} \overline{g_{a_j}})) = (\cap_{a_i \in A_{s_1} \cap A_{s_2}} g_{a_i}) \cap (\cap_{a_j \in A_{t_1} \cap A_{t_2}} \overline{g_{a_j}}) = g^I_{A_{s_1} \cap A_{s_2}, A_{t_1} \cap A_{t_2}}$.

(2) It can be easily proved by the structure of $g^I_{A_s,A_t}$. □

According to Proposition 2.1, for two information granules $g^I_{A_{s_1},A_{t_1}}$ and $g^I_{A_{s_2},A_{t_2}}$, we can get a meaningful equation: $g^I_{A_{s_1},A_{t_1}} \cap g^I_{A_{s_2},A_{t_2}} = g^I_{A_{s_1}\cap A_{s_2},A_{t_1}\cap A_{t_2}}$. It means that $g^I_{A_{s_1},A_{t_1}} \cap g^I_{A_{s_2},A_{t_2}}$ is the set of all objects with attributes in $A_{s_1} \cap A_{s_2}$ but without attributes in $A_{t_1} \cap A_{t_2}$.

Furthermore, based on the information granule $g^I_{A_s,A_t}$, a granular structure $\mathcal{G}^I$ can be explored, i.e.,

$$\mathcal{G}^I = \{g^I_{A_s,A_t}(\neq \emptyset) | \emptyset \subseteq A_s, A_t \subseteq A, A_s \cap A_t = \emptyset, A_s \cup A_t \neq \emptyset\}.$$

Suppose that $I = (U, A, \{V_{a_i}|a_i \in A\}, \{f_{a_i}|a_i \in A\})$ is an information table. Based on the characteristics of $g^I_{A_s,A_t}$, there are several explanations about the granular structure $\mathcal{G}^I$ as follows.

(1) First, if $\emptyset \subseteq A_s, A_t \subseteq A, A_s \cap A_t = \emptyset, A_s \cup A_t = AT$, it is clear that

$$\mathcal{G}^I = \{g^I_{A_s,A_t} | \emptyset \subseteq A_s, A_t \subseteq A, A_s \cap A_t = \emptyset, A_s \cup A_t = AT\}$$

is a partition of OB. In 1982, Pawlak developed two operators named upper and lower approximations by using a partition of the universe and then proposed the rough set theory [59].

(2) Second, if $A_s = \{a_i\}, A_t = \emptyset$, then $g^I_{A_s,A_t} = g^I_{\{a_i\},\emptyset} = g_{a_i}$. Thus,

$$\mathcal{G}^I = \{g_{a_1}, g_{a_2}, \cdots, g_{a_6}\}$$

is a covering of OB. In 1983, Zakowski developed two operators named upper and lower approximations by using a covering of the universe and then generalized the rough set theory [60]. At present, covering rough set is still a hot topic in rough set theory.

(3) Third, if A^j_s is a subset of AT such that for each $a \in A^j_s$, one can find that $o_j \in g_a$, and for each $a \in A/A^j_s$, we have that $o_j \overline{\in} g_a$. Then

$$g^I_{A^j_s,\emptyset} = \cap\{g_a | a \in A^j_s\}$$

is the intersection of all sets including o_j in granular structure $\{g_{a_1}, g_{a_2}, \cdots, g_{a_6}\}$. $g^I_{A^j_s,\emptyset}$ is called the neighborhood of o_j with respect to the granular structure $\{g_{a_1}, g_{a_2}, \cdots, g_{a_6}\}$. Meanwhile,

$$\mathcal{G}^I = \{g^I_{A^1_s,\emptyset}, g^I_{A^2_s,\emptyset}, \cdots, g^I_{A^n_s,\emptyset}\}$$

is called the neighborhood covering with respect to the granular structure $\{g_{a_1}, g_{a_2}, \cdots, g_{a_6}\}$ [61]. Later, lot of scholars further generalized the concept of neighborhood and got many excellent results about rough sets of neighborhood covering[62–64].

(4) Finally, based on the above analysis, one can find that

$$\mathcal{G}^I = \{g^I_{A_s,A_t}(\neq \emptyset) | \emptyset \subseteq A_s, A_t \subseteq A, A_s \cap A_t = \emptyset, A_s \cup A_t \neq \emptyset\}$$

is still a covering of OB and contains richer and more complex information granules. At the same time, according to the structure of $g^I_{A_s,A_t}$, a more effective granular computing model can be proposed in the third section of this paper.

3. A novel granular computing model based on three-way decision

In this section, based on the granules and granular structures shown in Section 2, we propose a granular computing model. To make it easier to understand and describe, some concrete meanings of the concepts related to the model first given. In addition, we also discuss the abstract computing properties of many concepts of the model. These concrete meanings and abstract properties play an important and irreplaceable role in the understanding and the application of this model.

Definition 3.1. Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table and $A_s, A_t \subseteq AT$, where $A_s \cap A_t = \emptyset$, and $A_s \cup A_t \neq \emptyset$. For each $X \subseteq OB$,

$$OP^I_{A_s,A_t}(X) = \{o \in OB | o \in g^I_{A_s,A_t} \cap X\}$$

is called the description set of X with respect to A_s, A_t. Meanwhile, $OP^I_{A_s,A_t}$ is called the description operator with respect to A_s, A_t.

Obviously, $OP^I_{A_s,A_t}(X)$ is a set of the objects in X that have attributes in A_s but do not have attributes in A_t. For each $X \subseteq OB$, if $A_s \cap A_t \neq \emptyset$ or $A_s \cup A_t = \emptyset$, then $OP^I_{A_s,A_t}(X) = \emptyset$. So in Definition 3.1, A_s, A_t must meet the conditions: $A_s \cap A_t = \emptyset$ and $A_s \cup A_t \neq \emptyset$. In addition, the description set $OP^I_{A_s,A_t}(X)$ not only has a clear meaning, but also plays an

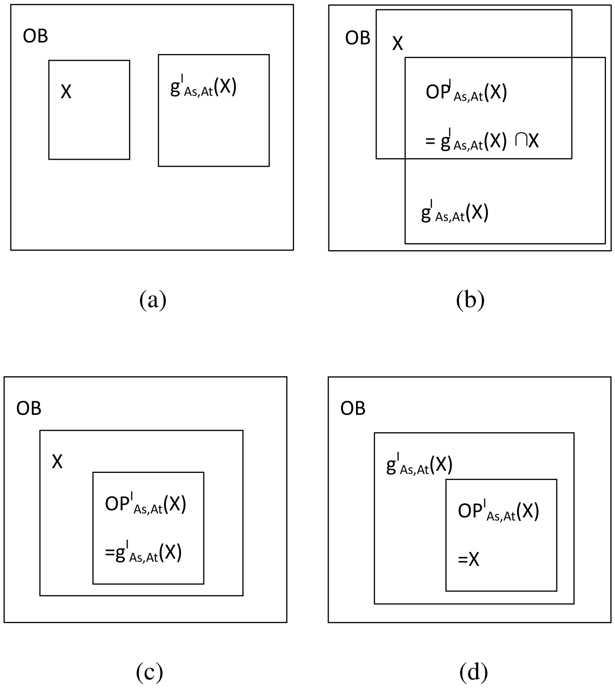

Fig. 1. Four diagrams of Proposition 3.1.

important role in the following research of this paper. Therefore, for each $X \subseteq OB$, based on the structure of $OP^I_{A_s,A_t}$, we develop a heuristic algorithm for computing $OP^I_{A_s,A_t}(X)$ in Section 5.

To better understand what the Definition 3.1 means, two propositions about the concrete meanings of $OP^I_{A_s}(X)$ are given as follows.

Proposition 3.1. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For each $X \subseteq OB$, the following results hold.*

(1) $OP^I_{A_s,A_t}(X) = \emptyset$ means that each object with attributes in A_s but without attributes in A_t does not belong to X;

(2) $OP^I_{A_s,A_t}(X) \neq \emptyset$, $OP^I_{A_s,A_t}(X) \neq g^I_{A_s,A_t}$ and $OP^I_{A_s,A_t}(X) \neq X$ means that only part of the objects with attributes in A_s but without attributes in A_t belongs to X;

(3) $OP^I_{A_s,A_t}(X) = g^I_{A_s,A_t}$ means that all objects with attributes in A_s but without attributes in A_t belong to X;

(4) $OP^I_{A_s,A_t}(X) = X$ means that each object in X has the attributes in A_s but does not have the attributes in A_t.

Based on the meanings of Proposition 3.1, we will raise a question: how do we tell if all the objects with attributes in A_s but without attributes in A_t belong to X? According to Proposition 3.1, we know that if $OP^I_{A_s,A_t}(X) = g^I_{A_s,A_t}$, then all objects with attributes in A_s but without attributes in A_t belong to X. If $\emptyset \neq OP^I_{A_s,A_t}(X) \neq g^I_{A_s,A_t}$, then there are some objects with attributes in A_s but without attributes in A_t that do not belong to X.

Next, in order to understand Proposition 3.1 more intuitively and visually, four diagrams in Fig. 1 are shown to further explain the four conclusions in Proposition 3.1, and the four results (1)-(4) in property 3.1 correspond to the four diagrams (a)-(d) in Fig. 1, respectively.

Similarly, we will also ask how many objects with attributes in A_s but without attributes in A_t are included in OB. The following result gives a good answer to this question.

Proposition 3.2. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. Then the following results hold.*

(1) $OP^I_{A_s,A_t}(OB) = \emptyset$ means there is no object in OB that has attributes in A_s but does not have attributes in A_t.

(2) $OP^I_{A_s,A_t}(OB) \neq \emptyset$ means there are objects in OB that has attributes in A_s but does not have attributes in A_t, and $g^I_{A_s,A_t}$ is the set of all objects in OB that have attributes in A_s but do not have attributes in A_t.

As a special case of Propositions 3.1 and 3.2, how do we tell if all the objects with only attributes in A_s belong to X or OB? Next, let's answer the question.

Corollary 3.1. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For $\emptyset \neq A_s \subseteq AT$ and $X \subseteq OB$, we have the following results.*

(1) $OP^I_{A_s,\emptyset}(X) = \emptyset$ means that each object with only attributes in A_s does not belong to X;

(2) $\emptyset \neq OP^I_{A_s,\emptyset}(X) \neq g^I_{A_s,\emptyset}$ means that part of the objects with only attributes in A_s belongs to X;

(3) $OP^I_{A_s,\emptyset}(X) = g^I_{A_s,\emptyset}$ means that all objects with only attributes in A_s belong to X.

(4) $OP^I_{A_s,\emptyset}(OB) = \emptyset$ means there is no object in OB that has attributes in A_s but does not have attributes in A_t.

(5) $OP^I_{A_s,\emptyset}(OB) \neq \emptyset$ means there are objects in OB that has attributes in A_s but does not have attributes in A_t, and $g^I_{A_s,\emptyset}$ is the set of all objects in OB that have attributes in A_s but do not have attributes in A_t.

In the above, we first introduce the specific meanings of the model in different situations, which are helpful for our intuitive understanding of the model. Next, we will discuss some basic mathematical properties of the model from the perspective of computing, which will greatly promote our abstract understanding of the model.

Proposition 3.3. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For each $X \subseteq OB$, the following results hold.*

(1) $OP^I_{A_s,A_t}(\emptyset) = \emptyset$;

(2) $OP^I_{A_s,A_t}(X) \subseteq X$;

(3) $OP^I_{A_s,A_t}(OB) = g^I_{A_s,A_t}$;

(4) $OP^I_{A_s,A_t}(OP^I_{A_s,A_t}(X)) = OP^I_{A_s,A_t}(X)$;

Proof. It is immediate from Definition 3.1. □

Proposition 3.4. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For $X, Y \subseteq OB$, the following results hold.*

(1) $X \subseteq Y \Rightarrow OP^I_{A_s,A_t}(X) \subseteq OP^I_{A_s,A_t}(Y)$;

(2) $OP^I_{A_s,A_t}(X \cap Y) = OP^I_{A_s,A_t}(X) \cap OP^I_{A_s,A_t}(Y)$;

(3) $OP^I_{A_s,A_t}(X \cup Y) = OP^I_{A_s,A_t}(X) \cup OP^I_{A_s,A_t}(Y)$;

Proof. (1) It is immediate by Definition 3.1.

(2) (⇒:) Clearly, the conclusion $OP^I_{A_s,A_t}(X \cap Y) \subseteq OP^I_{A_s,A_t}(X) \cap OP^I_{A_s,A_t}(Y)$ holds.

(⇐:) For each $o \in OP^I_{A_s,A_t}(X) \cap OP^I_{A_s,A_t}(Y)$, we have that $o \in OP^I_{A_s,A_t}(X)$ and $o \in OP^I_{A_s,A_t}(Y)$. By Definition 3.1, $o \in g^I_{A_s,A_t} \cap X$ and $o \in g^I_{A_s,A_t} \cap Y$, then $o \in g^I_{A_s,A_t} \cap (X \cap Y)$. Therefore, $o \in OP^I_{A_s,A_t}(X \cap Y)$.

(3) (⇒:) For each $o \in OP^I_{A_s,A_t}(X \cup Y)$, we have that $o \in g^I_{A_s,A_t} \cap (X \cup Y)$, then $o \in g^I_{A_s,A_t} \cap X$ or $o \in g^I_{A_s,A_t} \cap Y$. Hence, $o \in OP^I_{A_s,A_t}(X) \cup OP^I_{A_s,A_t}(Y)$.

(⇐:) The conclusion $OP^I_{A_s,A_t}(X \cup Y) \supseteq OP^I_{A_s,A_t}(X) \cup OP^I_{A_s,A_t}(Y)$ is clearly valid. □

According to Proposition 2.1, it is natural that the following result holds.

Proposition 3.5. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For any $X \subseteq OB$, then we have the following results.*

(1) $OP^I_{A_{s_1},A_{t_1}}(X) \cap OP^I_{A_{s_2},A_{t_2}}(X) = OP^I_{A_{s_1} \cap A_{s_2}, A_{t_1} \cap A_{t_2}}(X)$;

(2) $OP^I_{A_{s_1},A_{t_1}}(X) \cup OP^I_{A_{s_2},A_{t_2}}(X) \subseteq OP^I_{A_{s_1} \cup A_{s_2}, A_{t_1} \cup A_{t_2}}(X)$.

We know that $OP^I_{A_s,A_t}(X)$ is a set of the objects in X that have attributes in A_s but do not have attributes in A_t, while $OP^I_{A_s,A_t}(OB)$ is a set of the objects in OB that have attributes in A_s but do not have attributes in A_t. Usually there exists $x \in OB$ such that $x \in OP^I_{A_s,A_t}(OB)$, but $x \overline{\in} OP^I_{A_s,A_t}(X)$. Based on this fact, the following definition is proposed.

Definition 3.2. Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table and $A_s, A_t \subseteq A$, where $A_s \cap A_t = \emptyset$, and $A_s \cup A_t \neq \emptyset$. For each $X \subseteq OB$,

$$DD^I_{A_s,A_t}(X) = |OP^I_{A_s,A_t}(X)|/|OP^I_{A_s,A_t}(OB)|$$

is called the description degree of X with respect to A_s, A_t. If $|OP^I_{A_s,A_t}(OB)| = 0$, then we say that the description degree of X with respect to A_s, A_t does not exist.

For the description degree of X with respect to A_s, A_t, there are several explanations as follows:
(1) If $|OP^I_{A_s,A_t}(OB)| \neq 0$, then $0 \leq DD^I_{A_s,A_t}(X) \leq 1$.
(2) Because of $OP^I_{A_s,A_t}(OB) = g^I_{A_s,A_t}$, $DD^I_{A_s,A_t}(X)$ can also be written as $|OP^I_{A_s,A_t}(X)|/|g^I_{A_s,A_t}|$.
(3) If $\mathcal{P} = \{X_1, X_2, \cdots, X_p\}$ is a partition of OB, then $DD^I_{A_s,A_t}(X_1) + DD^I_{A_s,A_t}(X_2) + \cdots + DD^I_{A_s,A_t}(X_p) = 1$.

Example 3.1 *(Continued from Example 2.1).* For $A_s = \{a_4\}, A_t = \{a_3, a_5\}$, and $X = \{o_1, o_2, o_3, o_4\}$, then we have that $OP^I_{A_s,A_t}(OB) = \{o_2, o_6, o_7, o_8\}$ and $OP^I_{A_s,A_t}(X) = \{o_2\}$. Based on Definition 3.2, then $DD^I_{A_s,A_t}(X) = 0.25$.

Attribute reduction is a very important and basic content in granular computing theory [42–44,65,66]. When we analyze the data, one can find that some attributes in the information system do not have any influence on the granular structures or the granular computing models. In this case, it is necessary to eliminate these redundant attributes. According to the structural characteristics of the information granules proposed in Section 2, a new attribute reduction is introduced as follows:

Definition 3.3. Let $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ be an information table. For any $A_s, A_t \subseteq AT$, and $A_s \cap A_t = \emptyset$, if there exists $a_s \in A_s$ or $a_t \in A_t$ such that $g^I_{A_s/\{a_s\},A_t} = g^I_{A_s,A_t}$ or $g^I_{A_s,A_t/\{a_t\}} = g^I_{A_s,A_t}$, then a_s and a_t are respectively called the reducible attributes of A_s and A_t. Otherwise, a_s and a_t are respectively called the irreducible attributes of A_s and A_t. The ordered pair (A_s^{Red}, A_t^{Red}) is called a reduction of (A_s, A_k), if (A_s^{Red}, A_t^{Red}) satisfies two conditions: (1) $g^I_{A_s^{Red},A_t^{Red}} = g^I_{A_s,A_t}$; (2) For each $a'_s \in A_s^{Red}$, and each $a'_t \in A_t^{Red}$, one can find that $g^I_{A_s^{Red}/\{a'_s\},A_t^{Red}} \neq g^I_{A_s,A_t}$, and $g^I_{A_s^{Red},A_t^{Red}/\{a'_t\}} \neq g^I_{A_s,A_t}$.

Proposition 3.6. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table, and $A_s, A_t \subseteq AT$, where $A_s \cap A_t = \emptyset$, and $a_s \in A_s$, then for any $X \subseteq OB$, $OP^I_{A_s/\{a_s\},A_t}(X) = OP^I_{A_s,A_t}(X)$ if and only if a_s is a reducible attribute of A_s and A_t.*

Proof. It can be proved easily by Definitions 3.1 and 3.2. □

Let $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ be an information table, $A_s, A_t \subseteq AT$, where $A_s \cap A_t = \emptyset$ and $A_s \cup A_t \neq \emptyset$. First, suppose $A_{s_k} = A_s/\{a_{s_1}, a_{s_2}, \cdots, a_{s_k}\}$, $k = 1, 2, \cdots, p$. Obviously, for $a_{s_1} \in A_s$, $(\cap_{a_i \in A_{s_1}} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}}) = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$ if and only if a_{s_1} is a reducible attribute of A_s and A_t. Then, for any $X \subseteq OB$, from Proposition 3.6, one can find that $OP^I_{A_{s_1},A_t}(X) = OP^I_{A_s,A_t}(X)$. Similarly, for $a_{s_2} \in A_{s_1}$, $(\cap_{a_i \in A_{s_2}} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}}) = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$ if and only if a_{s_2} is a reducible attribute of A_{s_1} and A_t. Then we have that $OP^I_{A_{s_2},A_t}(X) = OP^I_{A_s,A_t}(X)$. Repeat the previous steps until there is a_{s_p} (where $a_{s_p} \in A_{s_{p-1}}$) satisfying the following two conditions: (1) $(\cap_{a_i \in A_{s_p}} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}}) = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$; (2) For each $a_{s_m} \in A_{s_p}$, $(\cap_{a_i \in A_{s_p}/\{a_{s_m}\}} g_{a_i}) \cap ((\cap_{a_j \in A_t} \overline{g_{a_j}})) \neq (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$. Hence, according to Proposition 3.6, we have that $OP^I_{A_{s_p}}(X) = OP^I_{A_s}(X)$.

Second, suppose $A_{t_l} = A_t/\{a_{t_1}, a_{t_2}, \cdots, a_{t_l}\}$, $l = 1, 2, \cdots, q$. For $a_{t_1} \in A_t$, $(\cap_{a_i \in A_{s_p}} g_{a_i}) \cap (\cap_{a_j \in A_{t_1}} \overline{g_{a_j}}) = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$ if and only if a_{t_1} is a reducible attribute of A_{s_p} and A_t. Then one can find that $OP^I_{A_{s_p},A_{t_1}}(X) = OP^I_{A_s,A_t}(X)$. Similarly, for $a_{t_2} \in A_{t_1}$, $(\cap_{a_i \in A_{s_p}} g_{a_i}) \cap (\cap_{a_j \in A_{t_2}} \overline{g_{a_j}}) = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$ if and only if a_{s_2} is a reducible attribute of A_{s_p} and A_{t_1}. Then we have that $OP^I_{A_{s_p},A_{t_2}}(X) = OP^I_{A_s,A_t}(X)$. Repeat the previous steps until there exists a_{t_q} (where $a_{t_q} \in A_{t_{q-1}}$) satisfying the following two conditions: (1) $(\cap_{a_i \in A_{s_p}} g_{a_i}) \cap (\cap_{a_j \in A_{t_q}} \overline{g_{a_j}}) = (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$; (2) For each $a_{t_n} \in A_{t_q}$, $(\cap_{a_i \in A_{s_p}} g_{a_i}) \cap (\cap_{a_j \in A_{t_q}/\{a_{t_n}\}} \overline{g_{a_j}}) \neq (\cap_{a_i \in A_s} g_{a_i}) \cap (\cap_{a_j \in A_t} \overline{g_{a_j}})$. Thus, (A_{s_p}, A_{t_q}) is a reduction of (A_s, A_t), and according to Proposition 3.6, we have that $OP^I_{A_{s_p},A_{t_q}}(X) = OP^I_{A_s,A_t}(X)$.

Corollary 3.2. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table, and $A_s, A_t \subseteq AT$, where $A_s \cap A_t = \emptyset$, $A'_s \subseteq A_s$, and $A'_t \subseteq A_t$, then for any $X \subseteq OB$, $OP^I_{A'_s,A'_t}(X) = OP^I_{A_s,A_t}(X)$ if and only if (A'_s, A'_t) is a reduction of (A_s, A_t).*

Let $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ be an information table, and $A_s, A_t \subseteq AT$, then a reduction of (A_s, A_t) can be obtained step by step according to the steps introduced above. Here, an example is employed to illustrate how to get the reduction of (A_s, A_t).

Example 3.2 *(Continued from Example 2.1).* Let $A_s = \{a_1, a_4, a_6\}$, $A_t = \{a_2, a_5\}$, we have that $(g_{a_1} \cap g_{a_4} \cap g_{a_6}) \cap (\overline{g_{a_2}} \cap \overline{g_{a_5}}) = \{o_5, o_7\}$. Now let's try to find a reduction of (A_s, A_t). First, $(g_{a_4} \cap g_{a_6}) \cap (\overline{g_{a_2}} \cap \overline{g_{a_5}}) = \{o_5, o_7\}$, then a_1 is an reducible attribute of A_s, and A_t. Second, it is clear that $g_{a_6} \cap (\overline{g_{a_2}} \cap \overline{g_{a_5}}) = \{o_5, o_7\}$, then a_4 is also a reducible attribute of A_s and A_t. However, $\overline{g_{a_2}} \cap \overline{g_{a_5}} = \{o_2, o_5, o_6, o_7\} \neq \{o_5, o_7\}$, so a_6 is an irreducible attribute of A_s, A_t. Similarly, based on the equation $g_{a_6} \cap \overline{g_{a_5}} = \{o_5, o_7\}$, a_4 is an reducible attribute of A_s, A_t. Finally, one can find that $g_{a_6} = \{o_5, o_7\}$, a_5 is a reducible attribute of A_s and A_t, too. Therefore, $(\{a_6, \emptyset\})$ is a reduction of (A_s, A_t).

Proposition 3.7. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table, $A'_s \subseteq A_s$ and $A'_t \subseteq A_t$, then the following results hold.*

(1) For any $X \subseteq OB$, we have that $OP^I_{A_s,A_t}(X) \subseteq OP^I_{A'_s,A'_t}(X)$;

(2) If $a_s \in A'_s$, $a_t \in A'_t$ are both the reducible attributes of A'_s and A'_t, then $a_s \in A'_s$, $a_t \in A'_t$ are both the reducible attributes of A_s and A_t.

Proof. (1) Since $A'_s \subseteq A_s$ and $A'_t \subseteq A_t$, then $g^I_{A_s,A_t} \subseteq g^I_{A'_s,A'_t}$. By Definition 3.1, for any $X \subseteq OB$, we have that $OP^I_{A_s,A_t}(X) \subseteq OP^I_{A'_s,A'_t}(X)$.

(2) Since $a_s \in A'_s$ is the reducible attribute of A'_s and A'_t, then one can find that $g^I_{A'_s/\{a_s\},A'_t} = g^I_{A'_s,A'_t}$. Then we have that $g^I_{A'_s/\{a_s\},A'_t} \cap g^I_{A_s/A'_s,A_t/A'_t} = g^I_{A'_s,A'_t} \cap g^I_{A_s/A'_s,A_t/A'_t}$, i.e., $g^I_{A_s/\{a_s\},A_t} = g^I_{A_s,A_t}$. Thus, $a_s \in A'_s$ is the reducible attribute of A_s and A_t. Similarly, it is clear that $a_t \in A'_t$ is the reducible attribute of A_s and A_t. □

Definition 3.4. Let $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ be an information table. For any $X \subseteq OB$ and any $A_s, A_t \subseteq AT$ (where $A_s \cap A_t = \emptyset$), if there exists $a_s \in A_s$ or $a_t \in A_t$ such that $OP^I_{A_s/\{a_s\},A_t}(X) = OP^I_{A_s,A_t}(X)$ or $OP^I_{A_s,A_t/\{a_t\}}(X) = OP^I_{A_s,A_t}(X)$, then a_s and a_t are respectively called the reducible attributes of A_s and A_t with respect to X. Otherwise, a_s and a_t are respectively called the irreducible attributes of A_s and A_t with respect to X.

The ordered pair (A_s^{RedX}, A_t^{RedX}) is called a reduction of (A_s, A_k) with respect to X, if (A_s^{RedX}, A_t^{RedX}) satisfies two conditions: (1) $OP^I_{A_s^{RedX},A_t^{RedX}}(X) = OP^I_{A_s,A_t}(X)$. (2) For each $a'_s \in A_s^{RedX}$ and each $a'_t \in A_t^{RedX}$, one can find that $OP^I_{A_s^{RedX}/\{a'_s\},A_t^{RedX}}(X) \neq OP^I_{A_s,A_t}(X)$ and $OP^I_{A_s^{RedX},A_t^{RedX}/\{a'_t\}}(X) \neq OP^I_{A_s,A_t}(X)$.

Based on Definitions 3.3 and 3.4, we have the following two explanations:

(1) In Definition 3.4, if $X = OB$, then the two reductions defined in Definitions 3.3 and 3.4 are the same;

(2) In Definition 3.3, $OP^I_{A_s^{Red},A_t^{Red}}(X) = OP^I_{A_s,A_t}(X)$ is true for any $X \subseteq OB$, while in Definition 3.4, $OP^I_{A_s^{RedX},A_t^{RedX}}(X) = OP^I_{A_s,A_t}(X)$ is true only for a given $X \subseteq OB$.

Next, a specific example will be employed to further explain the differences between Definitions 3.3 and 3.4 as follows.

Example 3.3 *(Continued from Example 2.1).* Suppose that $X = \{o_1, o_2, o_4, o_7\}$, $A_s = \{a_3, a_4\}$, and $A_t = \{a_5, a_6\}$. On the one hand, based on Definition 3.3, $(\{a_3\}, \{a_5, a_6\})$ is the reduction of $(\{a_3, a_4\}, \{a_5, a_6\})$.

On the other hand, according to Definition 3.4, $(\{a_3\}, \emptyset)$ is the reduction of $(\{a_3, a_4\}, \{a_5, a_6\})$ with respect to $X = \{o_1, o_2, o_4, o_7\}$.

Proposition 3.8. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For $X \subseteq OB$, the following results hold.*

(1) If $OP^I_{A_s,A_t}(X) = \emptyset$ if and only if $g^I_{A_s,A_t} \cap X = \emptyset$;

(2) If $OP^I_{A_s,A_t}(X) \neq \emptyset$, $OP^I_{A_s,A_t}(X) \neq g^I_{A_s,A_t}$, and $OP^I_{A_s,A_t}(X) \neq X$ if and only if $g^I_{A_s,A_t} \cap X \neq \emptyset$, $g^I_{A_s,A_t} \nsubseteq X$, and $X \nsubseteq g^I_{A_s,A_t}$;

(3) If $OP^I_{A_s,A_t}(X) = g^I_{A_s,A_t}$ if and only if $g^I_{A_s,A_t} \subseteq X$.

(4) If $OP^I_{A_s,A_t}(X) = X$ if and only if $X \subseteq g^I_{A_s,A_t}$.

Proof. It is immediate from Definition 3.1. □

Proposition 3.9. *Suppose that $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ is an information table. For each $X \subseteq OB$, there exist $OP^I_{A_{s_1},A_{t_1}}(X), OP^I_{A_{s_2},A_{t_2}}(X), \cdots, OP^I_{A_{s_u},A_{t_u}}(X)$ such that $X = \cup_{i=1}^{u} OP^I_{A_{s_i},A_{t_i}}(X)$.*

Example 3.4 *(Continued from Example 2.1).* For $X = \{o_1, o_3, o_4, o_5, o_7\}$, $A_{s_1} = \{a_2, a_3\}$, $A_{t_1} = \{a_1\}$, $A_{s_2} = \{a_3\}$, $A_{t_2} = \{a_1, a_2, a_4\}$, $A_{s_3} = \{a_2\}$, $A_{t_3} = \{a_4\}$, $A_{s_4} = \{a_6\}$, and $A_{t_4} = \emptyset$, we have that $OP^I_{A_{s_1},A_{t_1}}(X) = \{o_1\}$; $OP^I_{A_{s_2},A_{t_2}}(X) = \{o_3\}$; $OP^I_{A_{s_3},A_{t_3}}(X) = \{o_4\}$; $OP^I_{A_{s_4},A_{t_4}}(X) = \{o_5, o_7\}$. Thus, $X = OP^I_{A_{s_1},A_{t_1}}(X) \cup OP^I_{A_{s_2},A_{t_2}}(X) \cup OP^I_{A_{s_3},A_{t_3}}(X) \cup OP^I_{A_{s_4},A_{t_4}}(X)$. Therefore, X is the union of four sets $OP^I_{A_{s_1},A_{t_1}}(X)$, $OP^I_{A_{s_2},A_{t_2}}(X)$, $OP^I_{A_{s_3},A_{t_3}}(X)$ and $OP^I_{A_{s_4},A_{t_4}}(X)$.

In Definitions 3.3 and 3.4, two concepts of attribute reduction are given. Sometimes, the degree of attribute reduction needs to be considered. So the concept of reduction degree is further introduced as follows.

Definition 3.5. Let $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$ be an information table. For any $A_s, A_t \subseteq AT$, and $A_s \cap A_t = \emptyset$, if (A_s^{Red}, A_t^{Red}) is a reduction of (A_s, A_k), then

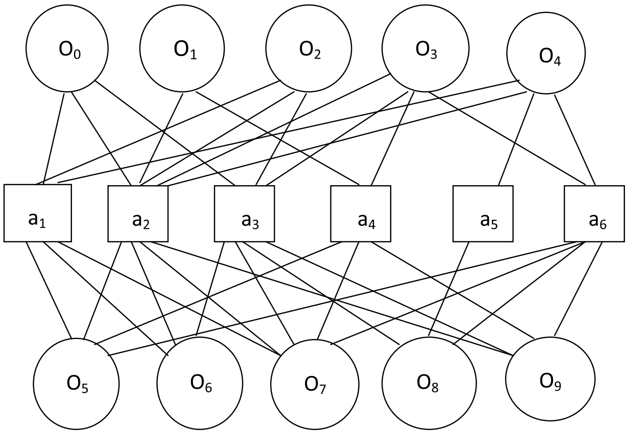

Fig. 2. An information security network.

$$RD^I(A_s^{Red}, A_t^{Red}) = 1 - (|A_s^{Red}| + |A_t^{Red}|)/(|A_s| + |A_t|)$$

is called the reduction degree of A_s, A_t.

For each $X \subseteq OB$, if $(A_s^{Red_X}, A_t^{Red_X})$ is the reduction of (A_s, A_k) with respect to X, then

$$RD_X^I(A_s^{Red_X}, A_t^{Red_X}) = 1 - (|A_s^{Red_X}| + |A_t^{Red_X}|)/(|A_s| + |A_t|)$$

is called the reduction degree of A_s, A_t with respect to X.

Obviously, $0 \le RD^I(A_s^{Red}, A_t^{Red}) \le 1 - 1/(|A_s| + |A_t|)$, and for each $X \subseteq OB$, $0 \le RD_X^I(A_s^{Red_X}, A_t^{Red_X}) \le 1 - 1/(|A_s| + |A_t|)$. Because A_s^{Red} and A_t^{Red} cannot both be empty sets, so $RD^I(A_s^{Red}, A_t^{Red}) \ne 1$. Similarly, $RD_X^I(A_s^{Red_X}, A_t^{Red_X}) \ne 1$.

What is the relationship between the two reduction degrees in Definition 3.5? The following conclusion answers this question.

Proposition 3.10. *Let $I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$ be an information table. For each $X \subseteq OB$ and any $A_s, A_t \subseteq AT$, where $A_s \cap A_t = \emptyset$, then $RD^I(A_s^{Red}, A_t^{Red}) \le RD_X^I(A_s^{Red_X}, A_t^{Red_X})$.*

Proof. For each $X \subseteq OB$, based on Definitions 3.3 and 3.4, one can find that $A_s^{Red_X} \subseteq A_s^{Red}$ and $A_t^{Red_X} \subseteq A_t^{Red}$. Then $|A_s^{Red_X}| + |A_t^{Red_X}| \le |A_s^{Red}| + |A_t^{Red}|$. Therefore, $1 - (|A_s^{Red}| + |A_t^{Red}|)/|A_s| + |A_t|) \le 1 - (|A_s^{Red_X}| + |A_t^{Red_X}|)/|A_s| + |A_t|)$. That is to say that $RD^I(A_s^{Red}, A_t^{Red}) \le RD_X^I(A_s^{Red_X}, A_t^{Red_X})$. □

Example 3.5 *(Continued from Example 3.3)*. For $X = \{o_1, o_2, o_4, o_7\}$, $A_s = \{a_3, a_4\}$, and $A_t = \{a_5, a_6\}$. One can find that $(A_s^{Red}, A_t^{Red}) = (\{a_3\}, \{a_5, a_6\})$. Then $RD^I(A_s^{Red}, A_t^{Red}) = 1 - 3/4 = 0.25$. On the other hand, because of $(A_s^{Red_X}, A_t^{Red_X}) = (\{a_3\}, \emptyset)$, so $RD_X^I(A_s^{Red_X}, A_t^{Red_X}) = 1 - 1/4 = 0.75$. Obviously, $RD^I(A_s^{Red}, A_t^{Red}) \le RD_X^I(A_s^{Red_X}, A_t^{Red_X})$.

4. An application in network security

Network security is now a hot research direction. With the increasing popularity of information technology, people not only enjoy the convenience brought by information technology, but also pay more attention to the security of information transmission. Therefore, it is particularly important to study various network security problems.

For example, an international flight taking off from the starting point needs to fly a long distance to reach the terminal point. When the aircraft is flying over the airspace of a country, the radar of that country will certainly verify the aircraft, and provide the altitude, heading and speed of the aircraft to other planes in the nearby airspace for the purpose of flight safety; If the aircraft does not fly over the airspace of a country, but passes near the airspace of that country, the radar of that country may or may not detect the aircraft; If the aircraft is far from the airspace of a country, the radar of that country will not identify the aircraft. In other words, during the flight of the aircraft, some radars need to detect it, some radars may or may not recognize it, and some radars will not verify it. So we can use the model based three-way decision proposed in this paper to analyze and deal with this aviation safety network.

Based on this example, if all the aircrafts and all the radars are respectively regarded as the information transceivers and the information detection stations, respectively, then an information security network similar to that shown in Fig. 2 can be induced.

As shown in Fig. 2, the information transceiver o_0 can send the information to detection station a_1. Then information detection station a_1 detects the information. Eventually, the detected information will be sent to information transceivers o_2, o_4, o_5, o_6, o_7. Meanwhile, the information transceiver o_0 can also send the information to detection station a_2. Then information detection station a_2 detects the information. Finally, the detected information will be sent to information transceivers $o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_9$.

Here, for the information security network shown in Fig. 2, the following basic assumptions need to be met:

(1) Any two information transceivers cannot share undetected information with each other;

(2) The information send and received by any transceiver should be detected by at least one information detection station.

Usually, an information security network can be defined by a tuple as follows:

$$\mathcal{INS} = (\mathcal{TC}, \mathcal{MS}, \{V_a | a \in \mathcal{MS}\}, \{f_a | a \in \mathcal{MS}\})$$

where $\mathcal{TC} = \{o_1, o_2, \cdots, o_n\}$ is a nonempty finite set of the information transceivers; $\mathcal{MS} = \{a_1, a_2, \cdots, a_m\}$ is a nonempty finite set of the information monitoring stations; $V = \cup_{a \in MS} V_a$, V_a is the domain of information monitoring station a; $f_a : \mathcal{TC} \to V_a$ is an information detection function. We use $f_a(o)$ to denote the value of the information transceiver o on the information monitoring station a. Suppose that for each $a \in \mathcal{MS}$, $V_a = \{0, 1\}$, where $f_a(o) = 1$ means that the information sent and transferred by o needs to be tested by a; $f_a(o) = 0$ means that the information sent and transferred by o does not need to be tested by a. For any $M_s, M_t \subseteq \mathcal{MS}$, T_{M_s,M_t} is a set of the information transceivers, in which all information transmitted and received by the information transceivers in T_{M_s,M_t} is detected by the monitoring stations in M_s, but not by the monitoring stations in M_t.

For different security networks, people's focus and problems to be solved are naturally different [56,57,67]. In order to better explain how to use the model introduced in this paper to deal with the network security presented in Fig. 2, here, the following definitions need to be proposed first.

Definition 4.1. Suppose that $\mathcal{INS} = (\mathcal{TC}, \mathcal{MS}, \{V_a | a \in \mathcal{MS}\}, \{f_a | a \in \mathcal{MS}\})$ is an information security network. For any $M_s, M_t \subseteq \mathcal{MS}$ and any $T \subseteq \mathcal{TC}$, then

$$\alpha^T_{M_s,M_t} = |T \cap T_{M_s,M_t}| / |T_{M_s,M_t}|$$

is called the inclusion rate of T with respect to M_s and M_t.

$\alpha^T_{M_s,M_t}$ is the ratio of the cardinalities of the two sets $T \cap T_{M_s,M_t}$ and T_{M_s,M_t}. Obviously, $0 \le \alpha^T_{M_s,M_t} \le 1$.

Definition 4.2. Suppose that $\mathcal{INS} = (\mathcal{TC}, \mathcal{MS}, \{V_a | a \in \mathcal{MS}\}, \{f_a | a \in \mathcal{MS}\})$ is an information security network. For any $M_{s_1}, M_{s_2}, M_{t_1}, M_{t_2} \subseteq \mathcal{MS}$, where $M_{s_1} \subseteq M_{s_2}$, and $M_{t_1} \subseteq M_{t_2}$. If $T_{M_{s_1},M_{t_1}} = T_{M_{s_2},M_{t_2}}$, then

$$\beta_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})} = 1 - (|M_{s_1}| + |M_{t_1}|)/(|M_{s_2}| + |M_{t_2}|)$$

is called the removal rate of M_{s_2}, M_{t_2} with respect to M_{s_1}, M_{t_1}.

For any $M_{s_1}, M_{s_2}, M_{t_1}, M_{t_2} \subseteq \mathcal{MS}$, (where $M_{s_1} \subseteq M_{s_2}$, and $M_{t_1} \subseteq M_{t_2}$), and any $T \subseteq \mathcal{TC}$. If $T_{M_{s_1},M_{t_1}} \cap T = T_{M_{s_2},M_{t_2}} \cap T$, then

$$\beta^T_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})} = 1 - (|M_{s_1}| + |M_{t_1}|)/(|M_{s_2}| + |M_{t_2}|)$$

is called the removal rate of M_{s_2}, M_{t_2} with respect to M_{s_1}, M_{t_1} and T.

$\beta_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})}$ (or $\beta^T_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})}$) can be seen as the quotient of the cardinalities of $M_{s_2} + M_{t_2} - M_{s_1} - M_{t_1}$ and $M_{s_2} + M_{t_2}$ while $T_{M_{s_2},M_{t_2}}$ (or $T_{M_{s_2},M_{t_2}} \cap T$) stays unchanged. The larger the $\beta_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})}$ is, the larger the $(|M_{s_2}| + |M_{t_2}|) - (|M_{s_1}| + |M_{t_1}|)$ is, so the more monitoring stations are removed from M_{s_2} and M_{t_2}.

For an information network, people usually pay attention to many problems, such as information transmission, information supervision, network stability and so on. Therefore, in the information network shown in Fig. 2, we also need to study and answer the following questions.

(Q_1) Which of the five information transceivers in $T = \{o_1, o_3, o_4, o_7, o_8\}$ sends and receives information only through information detection stations in $M_s = \{a_1, a_2, a_3\}$?

(Q_2) Which of all information transceivers sends and receives information only through information detection stations $M_s = \{a_1, a_2, a_3\}$?

(Q_3) For $T = \{o_1, o_3, o_4, o_7, o_8\}$, $M_s = \{a_1, a_2, a_3\}$ and $M_t = \emptyset$, what is the inclusion rate of T with respect to M_s and M_t?

(Q_4) Which of all information transceivers sends and receives information through information detection stations $M_s = \{a_2, a_6\}$ but not through $M_t = \{a_5\}$?

(Q_5) If the information detection station a_2 is under maintenance and cannot work properly, what impact will it have on the conclusion of questions 3 listed above?

Table 2
An information table based the information security.

OB	a_1	a_2	a_3	a_4	a_5	a_6
o_0	1	1	1	0	0	0
o_1	0	1	0	1	0	0
o_2	1	1	1	0	0	0
o_3	0	1	1	1	0	1
o_4	1	1	0	0	1	1
o_5	1	1	0	1	0	1
o_6	1	1	1	0	0	0
o_7	1	1	1	1	0	1
o_8	0	0	1	0	1	1
o_9	0	1	1	1	0	1

(Q_6) Which of the five information transceivers $T = \{o_0, o_1, o_2, o_5, o_7\}$ sends and receives information through information detection stations $M_s = \{a_1, a_2, a_6\}$ but not through $M_t = \{a_3, a_5\}$?

(Q_7) If the information detection station a_2 is under maintenance and cannot work properly, what impact will it have on the conclusion of questions 5 listed above?

(Q_8) For $M_{s_2} = \{a_1, a_2\}$, $M_{t_2} = \{a_4, a_5\}$,$M_{s_1} = \{a_1\}$ and $M_{t_1} = \{a_4\}$, then what is the removal rate of M_{s_2}, M_{t_2} with respect to M_{s_1}, M_{t_1}? For $T = \{o_1, o_2, o_6, o_9\}$, $M'_{s_1} = \emptyset$ and $M'_{t_1} = \{a_4\}$, then what is the removal rate of M_{s_2}, M_{t_2} with respect to M'_{s_1}, M'_{t_1} and T?

Next, we can use the model based on the three-way decision idea proposed in this paper to deal with the information security network shown in Fig. 2, and answer the above eight questions in four steps.

Step 1: $\mathcal{INS} = (\mathcal{TC}, \mathcal{MS}, \{V_a|a \in \mathcal{MS}\}, \{f_a|a \in \mathcal{MS}\})$ is the information network shown in Fig. 2, where $\mathcal{TC} = \{o_0, o_1, \cdots, o_9\}$ is the set of ten information transceivers, and $\mathcal{MS} = \{a_1, a_2, \cdots, a_6\}$ is the set of six information detection stations.

Step 2: According to Step 1, we can turn the information network to an information table $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$, where $OB = \mathcal{TC} = \{o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9\}$; $AT = \mathcal{MS} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$; $V = \cup_{a_j \in AT} V_{a_j}$, $V_{a_j} = \{0, 1\}$; $f_{a_j}(o_i) = 1$ means that the information sent by the information transceiver o_i will be checked by the information detection station a_j; $f_{a_j}(o_i) = 0$ means that the information sent by the information transceiver o_i won't be checked by the information detection station a_j, $i = 0, 1, \cdots, 9$, $j = 1, 2, \cdots, 6$. More details can be found in Table 2.

Step 3: Based on the information table presented in Table 2, twelve basic information granules can be respectively shown as follows.

$g_{a_1} = \{o_0, o_2, o_4, o_5, o_6, o_7\}$, $g_{a_2} = \{o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_9\}$, $g_{a_3} = \{o_0, o_2, o_3, o_6, o_7, o_8, o_9\}$, $g_{a_4} = \{o_1, o_3, o_5, o_7, o_9\}$, $g_{a_5} = \{o_4, o_8\}$, $g_{a_6} = \{o_3, o_4, o_5, o_7, o_8, o_9\}$;

$\overline{g_{a_1}} = \{o_1, o_3, o_8, o_9\}$, $\overline{g_{a_2}} = \{o_8\}$, $\overline{g_{a_3}} = \{o_1, o_4, o_5\}$, $\overline{g_{a_4}} = \{o_0, o_2, o_4, o_6, o_8\}$, $\overline{g_{a_5}} = \{o_0, o_1, o_2, o_3, o_5, o_6, o_7, o_9\}$, $\overline{g_{a_6}} = \{o_0, o_1, o_2, o_6\}$.

Based on the structures of g_{a_i} and $\overline{g_{a_i}}$, g_{a_i} is the set of all the information transceivers that the information sent and received must be detected by the information detection station a_i; and $\overline{g_{a_i}}$ is the set of all the information transceivers that the information sent and received do not need to be detected by the information detection station a_i, $i = 1, 2, 3, 4, 5, 6$.

Step 4: First, let's answer the first question. Let $X_1 = T = \{o_1, o_3, o_4, o_7, o_8\}$, $A_{s_1} = M_s = \{a_1, a_2, a_3\}$, $A_{t_1} = \{a_4, a_5, a_6\}$, then $g^I_{A_{s_1}, A_{t_1}} = \{o_0, o_2, o_6\}$. According to Definition 3.1, we have that $OP^I_{A_{s_1}, A_{t_1}}(X_1) = \emptyset$. From Proposition 3.1, we know that among the five information transceivers o_1, o_3, o_4, o_7, o_8, then there is no information transceiver in T that can send and receive information only through information detection stations in $M_s = \{a_1, a_2, a_3\}$.

Second, we will answer the second question. Let $X_2 = OB$, $A_{s_2} = M_s = \{a_1, a_2, a_3\}$, $A_{t_2} = \{a_4, a_5, a_6\}$, then $g^I_{A_{s_2}, A_{t_2}} = \{o_0, o_2, o_6\}$. According to Definition 3.1, we have that $OP^I_{A_{s_2}, A_{t_2}}(X_2) = g^I_{A_{s_2}, A_{t_2}} = \{o_0, o_2, o_6\}$. From Proposition 3.2, we know that among all the information transceivers, these three information transceivers o_0, o_2, o_6 can send and receive information only through information detection stations in $M_s = \{a_1, a_2, a_3\}$.

Third, we will answer the third question. Based on the relationship between the information security network shown in Fig. 2 and the information system presented in Table 2, if $T = X$, $M_s = A_s$, and $M_t = A_t$, then $\alpha^T_{M_s, M_t} = DD^I_{A_s, A_t}(X)$. According to Definition 3.2, one can find that $DD^I_{A_s, A_t}(X) = 0$. That is to say that the inclusion rate of T with respect to M_s and M_t is zero.

Fourth, let's answer the fourth question. Let $X_3 = OB$, $A_{s_3} = M_s = \{a_2, a_6\}$, $A_{t_3} = M_t = \{a_5\}$, we have that $g^I_{A_{s_3}, A_{t_3}} = \{o_3, o_5, o_7, o_9\}$. Thus, we have that $OP^I_{A_{s_3}, A_{t_3}}(X_3) = \{o_3, o_5, o_7, o_9\}$. Therefore, based on Proposition 3.2, one can find that information transceivers o_3, o_5, o_7, o_9 send and receive information through information detection stations in $M_s = \{a_2, a_6\}$ but not through information detection stations in $M_t = \{a_5\}$ in this information security network.

Fifth, let's answer the fifth question. According to Definition 3.4, a_2 is a reducible attribute of A_{s_3} and A_{t_3} with respect to X_3. Then, if the information detection station a_2 does not work, it won't make any difference to the conclusion of question 3.

Sixth, let's deal with the sixth question. Let $X_4 = T = \{o_0, o_1, o_2, o_5, o_7\}$, $A_{s_4} = M_s = \{a_1, a_2, a_6\}$, $A_{t_4} = M_t = \{a_3, a_5\}$, we have that $g^I_{A_{s_4},A_{t_4}} = \{o_5\}$. Thus, we have that $OP^I_{A_{s_4},A_{t_4}}(X_4) = \{o_5\}$. Therefore, based on Proposition 3.1, one can find that only information transceiver o_5 in T sends and receives information through information detection stations in $T = \{o_0, o_1, o_2, o_5, o_7\}$ but not through information detection stations in $M_s = \{a_1, a_2, a_6\}$ in this information security network.

Seventh, we will answer the seventh question. Since a_2 is a reducible attribute of A_{s_3} and A_{t_3}, $A_{s_3} \subseteq A_{s_4}$ and $A_{t_3} \subseteq A_{t_4}$, then based on Proposition 3.7, a_2 is also a reducible attribute of A_{s_4} and A_{t_4}. Therefore, if the information detection station a_2 does not work, it won't make any difference to the conclusion of question 6.

Finally, let's answer the last question. Let $M_{s_2} = A_s$, $M_{t_2} = A_t$, $M_{s_1} = A_s^{Red}$, $M_{t_1} = A_t^{Red}$, we have that $RD^I(A_s^{Red}, A_t^{Red}) = \beta_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})}$. So $\beta_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})} = 0.5$. Therefore, the removal rate of M_{s_2}, M_{t_2} with respect to M_{s_1}, M_{t_1} is 0.5. Similarly, let $X = T = \{o_1, o_2, o_6, o_9\}$, $M'_{s_1} = A_s^{Redx}$, $M'_{t_1} = A_t^{Redx}$, then $RD^I_X(A_s^{Redx}, A_t^{Redx}) = \beta^T_{(M_{s_2},M_{t_2}),(M_{s_1},M_{t_1})}$. Since $RD^I_X(A_s^{Redx}, A_t^{Redx}) = 0.75$, so the removal rate of M_{s_2}, M_{t_2} with respect to M'_{s_1}, M'_{t_1} and T is 0.75.

5. The algorithms for computing approximation set and the reduction

In this section, we design two algorithms for computing the description set, the description degree, the reduction and reduction degree. These two algorithms will play an important role in the future study of the applications of granular computing model proposed in this paper.

First, the Algorithm 1 is developed for computing the description set and description degree. The steps 1-2 compute the basic information granules based on the indispensable attributes. The steps 3-6 compute the information granule $g^I_{A_s,\emptyset}$ only related to A_t. Then, the information granule $g^I_{A_s,A_t}$ is computed by steps 7-11. We can get the description set $OP^I_{A_s,A_t}(X)$ of any subset $X \subseteq OB$ through the intersection of $g^I_{A_s,A_t}$ and X by step 12. Finally, steps 13-14 compute the description degree of X by the quotient of cardinal numbers of $g^I_{A_s,A_t} \cap X$ and $g^I_{A_s,A_t}$.

Algorithm 1: An algorithm for computing the description set $OP^I_{A_s,A_t}(X)$ and description degree $DD^I_{A_s,A_t}(X)$.

Input : An information system $I = (OB, AT, \{V_a|a \in AT\}, \{f_a|a \in AT\})$, two attribute sets $A_s = \{a_i^s|i = 1, 2\cdots, p\}$, $A_t = \{a_j^t|j = 1, 2, \cdots, q\}$, and $X \subseteq OB$;
Output : $OP^I_{A_s,A_t}(X)$ and $DD^I_{A_s,A_t}(X)$.
1 **begin**
2 Compute $g_{a_1^s}, g_{a_2^s}, \cdots, g_{a_p^s}$;
3 $g^I_{A_s,\emptyset} \leftarrow OB$;
4 **for** $i = 1 : p; i <= p; i + +$ **do**
5 $g^I_{A_s,\emptyset} \leftarrow g^I_{A_s,\emptyset} \cap g_{a_i^s}$
6 **end**
7 Compute $\overline{g_{a_1^t}}, \overline{g_{a_2^t}}, \cdots, \overline{g_{a_q^t}}$;
8 $g^I_{A_s,A_t} \leftarrow g^I_{A_s,\emptyset}$;
9 **for** $j = 1 : q; i <= q; j + +$ **do**
10 $g^I_{A_s,A_t} \leftarrow g^I_{A_s,A_t} \cap \overline{g_{a_j^t}}$
11 **end**
12 Compute $g^I_{A_s,A_t} \cap X$ // where $g^I_{A_s,A_t} \cap X = OP^I_{A_s,A_t}(X)$;
13 Compute $|g^I_{A_s,A_t} \cap X|/|g^I_{A_s,A_t}|$ // where $|g^I_{A_s,A_t} \cap X|/|g^I_{A_s,A_t}| = DD^I_{A_s,A_t}(X)$;
14 **end**

The time complexity analysis

The steps 1-2 compute the relationship between every object and attribute, and its time complexity is $|OB| * |AT|$. We consider each information granule which is induced by A_s in steps 3-6. So, the time complexity is $|A_s|$. In a similar reason, the time complexity is $|A_t|$ in steps 7-13. In summary, the time complexity of Algorithm 1 is $T(n) = |OB| * |AT| + |A_s| + |A_t|$. In addition, according to $|A_s| + |A_t| \leq |AT|$, then the time complexity is $O(|OB| * |AT|)$.

Second, Algorithm 2 is designed to compute the reduction and reduction degree related to Definitions 3.3, 3.4 and 3.5. The steps 1-3 can be obtained by Algorithm 1. The steps 4-8 and steps 9-13 are finding the reduction through keeping approximation set constant when an attribute is deleted from A_s and A_t. Finally, steps 14-15 compute the reduction degree of $X \subseteq OB$.

Table 3
The basic information of data sets.

No.	Data set name	Abbreviation	Objects	Attributes
1	*Autism − Child − Data*	ACD	292	10
2	*Autism − Adult − Data*	AAD	704	10
3	*SemeionHandwrittenDigit*	SHD	70	204
4	*Diabetes130 − UShospitals*	DUS	3983	20
5	*AbscisicAcidSignalingNetwork*	AASN	5456	43
6	*Sgemm Product*	SP	67360	4

Table 4
The time consumption of Algorithm 1 and Algorithm 2 when X take 10% objects.

Time		ACD		AAD		SHD		DUS		AASN		SP	
		$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$
A_s, A_t	10%	0.000	0.000	0.000	0.031	0.031	0.156	0.000	0.093	0.015	0.265	0.062	2.187
	20%	0.015	0.046	0.000	0.046	0.000	0.421	0.015	0.281	0.109	0.609	0.000	2.281
	30%	0.031	0.046	0.000	0.109	0.046	1.078	0.031	0.421	0.062	0.703	0.000	2.312
	40%	0.031	0.046	0.031	0.078	0.046	1.453	0.031	0.687	0.093	1.125	0.015	2.125
	50%	0.031	0.046	0.000	0.062	0.078	2.406	0.046	0.937	0.125	4.312	0.015	2.328

Algorithm 2: An algorithm for computing the reduct $(A_s^{Red_X}, A_t^{Red_X})$ and reduct degree $RD_X^I(A_s^{Red_X}, A_t^{Red_X})$.

Input : An information system $I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$, two attribute sets $A_s = \{a_i^s | i = 1, 2 \cdots, p\}$, $A_t = \{a_j^t | j = 1, 2, \cdots, q\}$ and $X \subseteq OB$;
Output : $A_s^{Red_X}, A_t^{Red_X}$ and $RD_X^I(A_s^{Red_X}, A_t^{Red_X})$.

1 **begin**
2 Compute $g_{a_1^s}, g_{a_2^s}, \cdots, g_{a_p^s}, \overline{g_{a_1^t}}, \overline{g_{a_2^t}}, \cdots, \overline{g_{a_q^t}}$;
3 Compute $g^I_{A_s, A_t}$;
4 **for** $i = 1 : p; i <= p; i++$ **do**
5 $A_s^{Red_X} \leftarrow A_s$;
6 If $g^I_{A_s/\{a_i^s\}, A_t} \cap X = g^I_{A_s, A_t} \cap X$ then
7 $A_s^{Red_X} \leftarrow A_s^{Red_X}/\{a_i^s\}$, $A_s \leftarrow A_s/\{a_i^s\}$
8 **end**
9 **for** $j = 1 : q; i <= q; j++$ **do**
10 $A_t^{Red_X} \leftarrow A_t$;
11 If $g^I_{A_s^{Red_X}, A_t/\{a_j^t\}} \cap X = g^I_{A_s^{Red_X}, A_t} \cap X$ then
12 $A_t^{Red_X} \leftarrow A_t^{Red_X}/\{a_j^t\}$, $A_t \leftarrow A_t/\{a_j^t\}$
13 **end**
14 Compute $1 - (|A_s^{Red_X}| + |A_t^{Red_X}|)/(|A_s| + |A_t|)$ // where $1 - (|A_s^{Red_X}| + |A_t^{Red_X}|)/(|A_s| + |A_t|) = RD_X^I(A_s^{Red_X}, A_t^{Red_X})$;
15 **end**

The time complexity analysis
The steps 1-3 compute the information granule $g^I_{A_s, A_t}$ and the time complexity is $(|A_s| + |A_s|) * |OB|$, where $|A_s| + |A_t| \leq |AT|$. The steps 4-15 compute the approximation set when an attribute is deleted in A_s and A_s. So, the time complexity is $(|A_s| + |A_s|) * |AT| * |OB|$, and $|A_s| + |A_t| \leq |AT|$. To sum up, the time complexity of Algorithm 2 is $T(n) = (|A_s| + |A_s|) * |OB| + (|A_s| + |A_s|) * |AT| * |OB|$. That is to say, the time complexity of Algorithm 2 is $O(|AT|^2 * |OB|)$.

6. Experimental analysis

In the previous section, we presented two algorithms for computing approximation set and the reduction, respectively. In this part, we choose six data sets in UCI for experimental analysis in order to verify the superiority and effectiveness of the algorithms. We pretreated the data in the experiment in order to ensure the validity of the experiment. We select the new data set composed of attributes whose attribute values are non metric data from the original data set which contain complex data types. This preprocessing method does not change the range of attribute values, which is beneficial to ensure the authenticity of the experiment. The basic information of data sets is shown in Table 3. These experiments are implemented by using Matlab R2016b and performed on a personal computer with an Intel Core i7-6700, 3.40 GHz CPU, 12.0 GB of memory, and 64-bit Windows 10.

We preprocess the data and only keep the 0-1 Boolean data set as the experimental data set. In order to study the general rules of approximation set and reduction and ensure the objectivity of the experiment, the experiment randomly selects some attributes from attribute sets A_s and A_t, and randomly selects some objects from object set X. Table 4 shows the time consumption of Algorithms 1 and 2 when X takes 10% of objects and A_s, A_t gradually increase. Table 5 shows

Table 5
The time consumption of Algorithm 1 and Algorithm 2 when X take 30% objects.

Time		ACD		AAD		SHD		DUS		AASN		SP	
		$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$
A_s, A_t	10%	0.000	0.000	0.000	0.046	0.000	0.109	0.000	0.125	0.015	0.171	0.000	6.531
	20%	0.000	0.015	0.000	0.031	0.046	0.421	0.015	0.218	0.031	0.531	0.015	6.734
	30%	0.000	0.015	0.000	0.046	0.031	0.781	0.078	0.515	0.046	1.250	0.062	6.921
	40%	0.000	0.031	0.000	0.078	0.031	1.359	0.046	0.984	0.140	2.281	0.000	6.875
	50%	0.000	0.031	0.031	0.093	0.031	2.390	0.093	1.078	0.125	3.062	0.000	6.796

Table 6
The time consumption of Algorithm 1 and Algorithm 2 when X take 50% objects.

Time		ACD		AAD		SHD		DUS		AASN		SP	
		$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$
A_s, A_t	10%	0.000	0.000	0.000	0.015	0.000	0.109	0.015	0.156	0.015	0.375	0.000	10.796
	20%	0.000	0.015	0.000	0.046	0.015	0.375	0.031	0.484	0.062	0.937	0.062	11.031
	30%	0.000	0.015	0.000	0.062	0.031	0.765	0.078	0.609	0.093	1.078	0.000	10.859
	40%	0.000	0.015	0.031	0.062	0.031	1.546	0.031	0.718	0.109	2.437	0.000	11.078
	50%	0.000	0.062	0.000	0.062	0.046	2.109	0.078	1.171	0.116	2.869	0.000	11.031

Table 7
The time consumption of Algorithm 1 and Algorithm 2 when X take 70% objects.

Time		ACD		AAD		SHD		DUS		AASN		SP	
		$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$	$AL1$	$AL2$
A_s, A_t	10%	0.000	0.000	0.000	0.031	0.046	0.171	0.015	0.125	0.015	0.328	0.062	15.500
	20%	0.000	0.000	0.031	0.031	0.000	0.359	0.062	0.203	0.093	0.750	0.062	15.453
	30%	0.000	0.015	0.000	0.046	0.031	0.734	0.031	0.500	0.093	1.296	0.062	15.593
	40%	0.000	0.015	0.015	0.046	0.015	1.359	0.031	0.750	0.109	2.656	0.015	15.843
	50%	0.000	0.015	0.031	0.093	0.031	2.015	0.046	1.031	0.140	2.906	0.000	15.484

Table 8
Description degree (DDI) of X when As,At takes 10% attributes and X take 10% objects for six data set.

	N1	N2	N3	N4	N5	N6	N7	N8	N9	N10	Average
ACD	0.042	0.054	0.128	0.066	0.047	0.021	0.238	0.150	0.060	0.106	0.091
AAD	0.086	0.067	0.048	0.098	0.076	0.152	0.112	0.120	0.117	0.096	0.097
SHD	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
DUS	0.000	0.095	0.000	0.000	0.121	0.135	0.181	0.091	0.086	0.000	0.071
AASN	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SP	0.099	0.099	0.100	0.102	0.100	0.1017	0.098	0.102	0.101	0.1027	0.100

Table 9
Reduction degree (RDI) of As,At when As,At takes 10% attributes and X take 10% objects for six data set.

	N1	N2	N3	N4	N5	N6	N7	N8	N9	N10	Average
ACD	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
AAD	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SHD	0.900	0.875	0.950	0.950	0.900	0.925	0.950	0.950	0.875	0.975	0.925
DUS	0.500	0.250	0.500	0.250	0.000	0.000	0.250	0.000	0.500	0.250	0.250
AASN	0.500	0.500	0.500	0.125	0.875	0.500	0.500	0.875	0.500	0.000	0.487
SP	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

the time consumption of Algorithms 1 and 2 when X takes 30% of objects and A_s, A_t gradually increase. Table 6 shows the time consumption of Algorithms 1 and 2 when X takes 50% of objects and A_s, A_t gradually increase. Table 7 shows the time consumption of Algorithms 1 and 2 when X takes 70% of objects and A_s, A_t gradually increase. Table 8 shows the description degree (DDI) of X when A_s, A_t takes 10% attributes and X take 10% objects for six data set. Table 9 shows the reduction degree (RDI) of A_s, A_t when A_s, A_t takes 10% attributes and X take 10% objects for six data set. Fig. 3 is a line chart which is showing the time consumption of Algorithm 1 changing with A_s, A_t when X takes different object sets. Fig. 4 is a line chart which is showing the time consumption of Algorithm 2 changing with A_s, A_t when X takes different object sets. Fig. 5 is a line chart which is showing the time consumption of Algorithm 1 changing with X when A_s and A_t take different attribute sets. Fig. 6 is a line chart which is showing the time consumption of Algorithm 2 changing with X when A_s and A_t take different attribute sets. Fig. 7 is a line chart which is showing the description degree (DDI) of X when

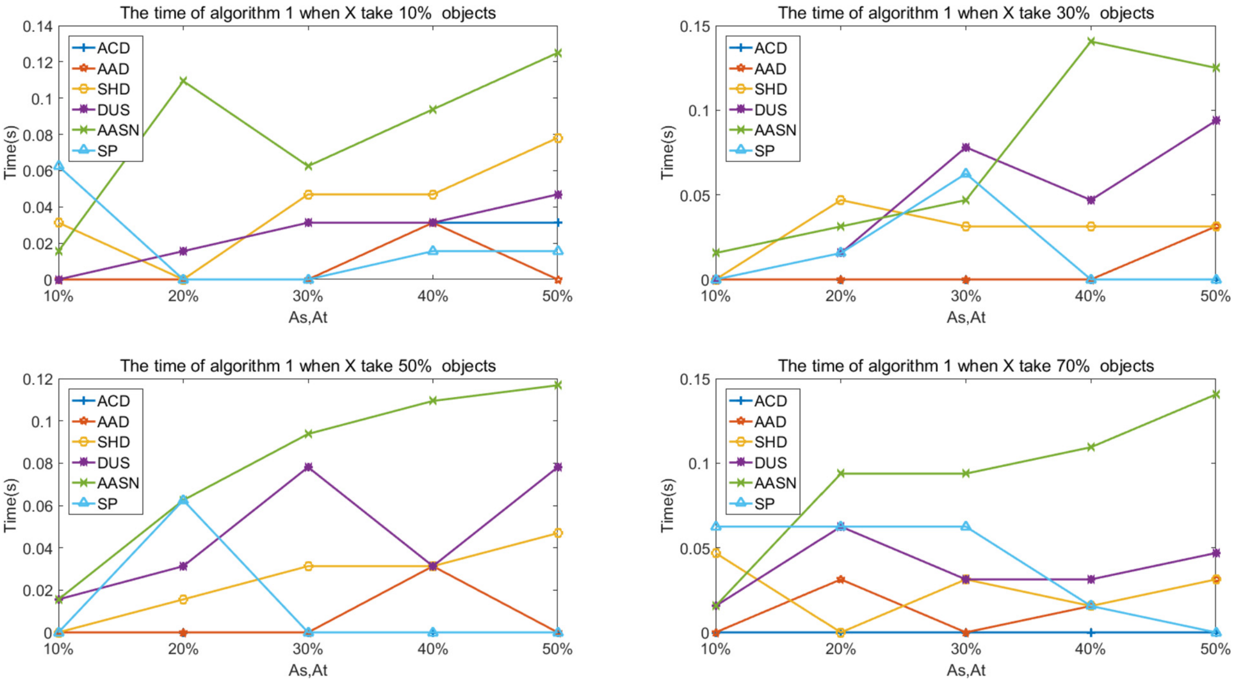

Fig. 3. The time consumption of Algorithm 1.

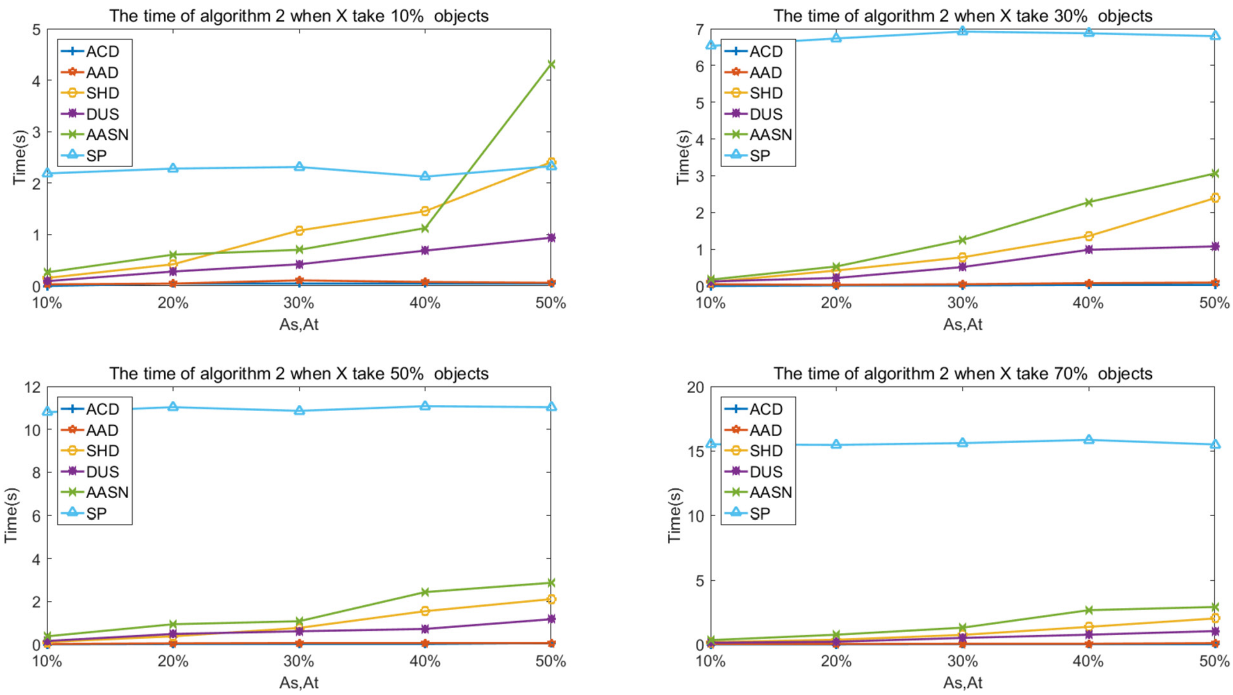

Fig. 4. The time consumption of Algorithm 2.

A_s, A_t takes 10% attributes and X take 10% objects for six data sets. Fig. 8 is a line chart which is showing the reduction degree (RDI) of A_s, A_t when A_s, A_t takes 10% attributes and X take 10% objects for six data sets. Fig. 9 is a line chart which is showing the average of DDI and RDI for six data sets. We can get the following conclusions by analyzing the experimental data.

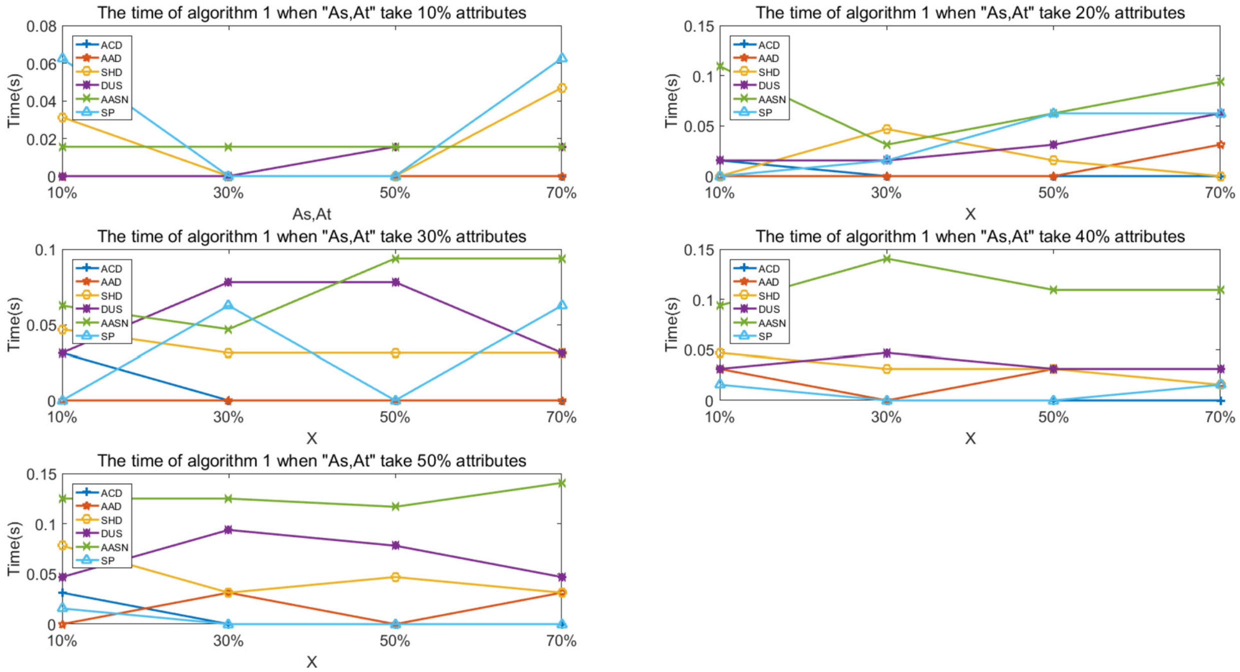

Fig. 5. The time consumption of Algorithm 1.

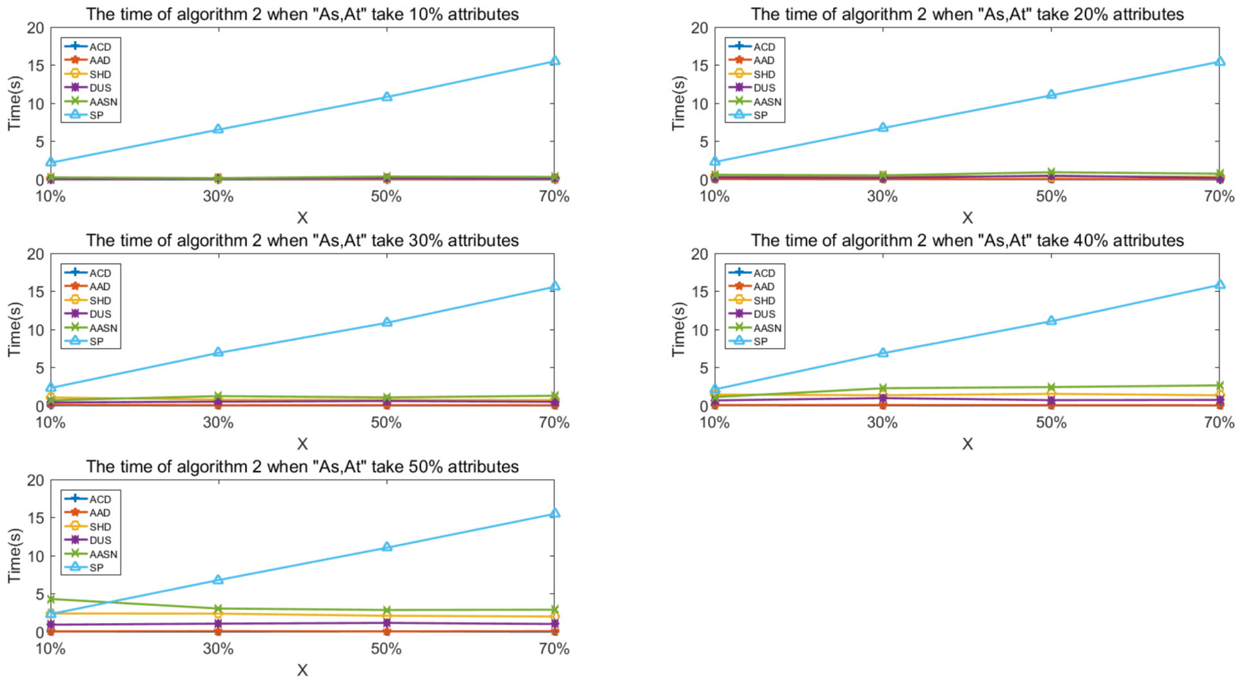

Fig. 6. The time consumption of Algorithm 2.

- The time consumption of Algorithm 1 is very small, even if the size of the data set is from tens to tens of thousands, the time consumption of Algorithm 1 is still less than 0.14 s. It shows that Algorithm 1 has very strong practical value and can be applied to high-dimensional data.
- The time consumption of Algorithm 2 is higher than that of Algorithm 1. But the time consumption is still controlled within 5 s, small time consumption is very important for the generalization of Algorithm 2. Algorithm 2 can quickly and efficiently eliminate redundant attributes from the data set and get reduction.

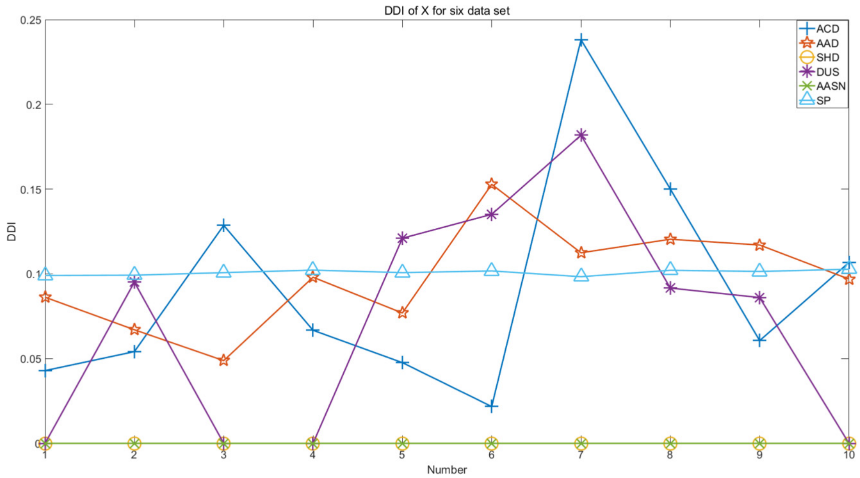

Fig. 7. DDI of X when As,At takes 10% attributes and X take 10% objects for six data set.

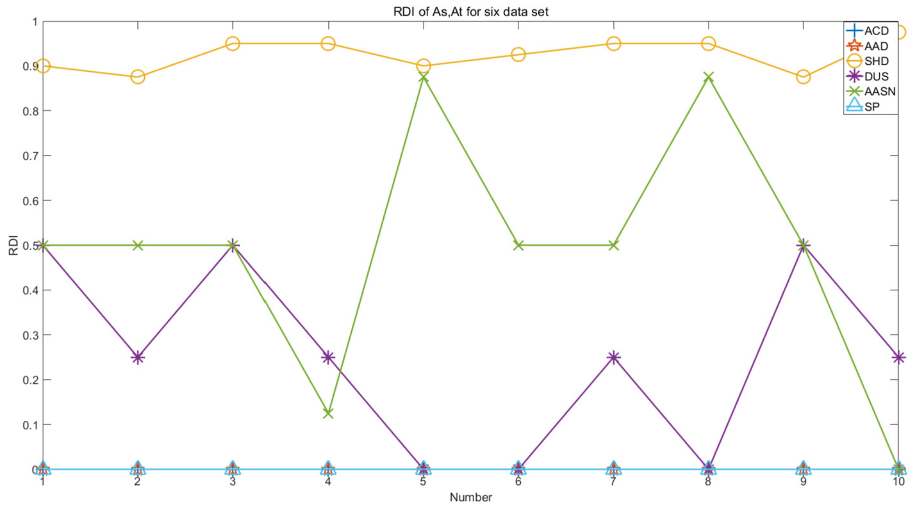

Fig. 8. RDI of As,At when As,At takes 10% attributes and X take 10% objects for six datasets.

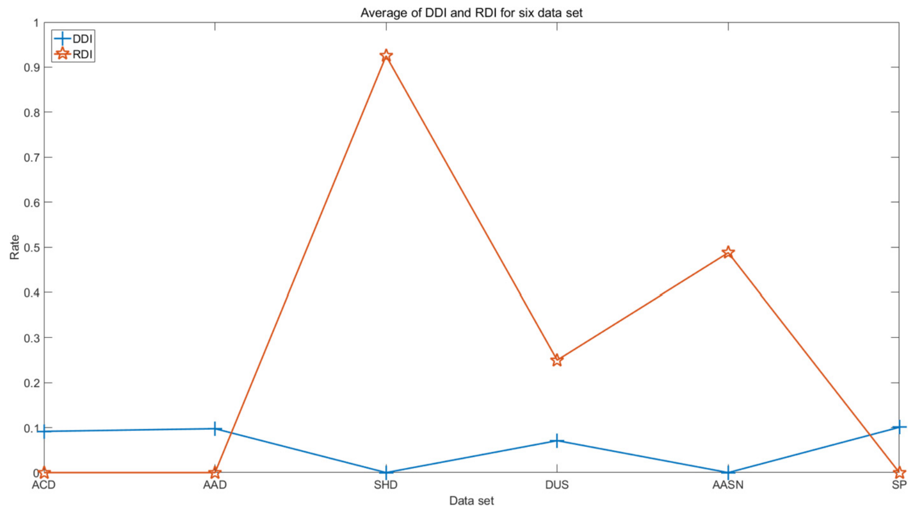

Fig. 9. Average of DDI and RDI for six data set.

• The time consumption of Algorithm 1 has no obvious rule for the change of X and A_s, A_t. Because Algorithm 1 needs to find every information granule, no matter how big A_s and A_t are.

• The time consumption of Algorithm 2 is increasing with A_s, A_t. Because Algorithm 2 needs to delete the elements from A_s, A_t one by one to determine whether the approximation set is the same. The more attributes in A_s, A_t, the more time consumption. Among them, the line graph of dataset "SP" is close to horizontal. Because "SP" has only four attributes, the number of 10% - 50% attributes is almost the same, and the time consumption is similar.

• The time consumption of Algorithms 1 and 2 has little relationship with the object set X.

• The description degree (DDI) of some data sets exists ($DDI > 0$), and the description degree (DDI) of some data does not exist ($DDI = 0$).

• The reduction degree (DDI) of data set SHD is close to 0.9 and the reduction degree (DDI) of data sets AASN, DUS is greater than 0. It means that reduction degree (DDI) has a good effect. However, not all data sets have good reduction effect.

In conclusion, the time consumption of Algorithms 1 and 2 is small, which is mainly related to the size of data set, but not to the object set. In particular, the size of attribute set has the greatest impact on Algorithm 2.

7. Conclusion

Granular computing method is an effective theory in the fields of data mining and knowledge discovery. Now, many scholars are committed to the research and generalization of granular computing theory. Since 2010, the three-way decision theory has gradually developed. Especially in recent years, the research on three-way decision theory has become more and more popular and fruitful. Therefore, it is necessary to combine granular computing theory with three-way idea to build new methods and models to solve more complex data problems.

In this paper, according to the idea of three-way decision, we construct a new class of information granules by dividing all attributes of data into three disjoint parts. Based on the information granules, a novel granular computing model is proposed. In terms of theory, on the one hand, we make a series of semantic interpretations of the information granules and the model, respectively. On the other hand, we also study and summarize many computational or mathematical properties of the model. In application, we illustrate that the model is suitable for dealing with network security problem. In order to facilitate the model to solve practical problems, we design the relevant algorithms. Through detailed numerical experiments, we find that the algorithms have ideal time consumption. Meanwhile, we also analyze the factors affecting the effectiveness of the algorithms.

In this paper, only a superficial study of this new granular structure is carried out, and lots of works need to be further discussed. For example, we can investigate the algebraical and topological properties of this model and explore its applications in other practical problems. In addition, according to the granular structure given in this paper, the relevant rough set models can also be established, and the theories and applications of the rough set models will be deeply researched in the near future.

Compliance with ethical standards

Research involving human participants and/or animals: This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent: Authors confirm that the article has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. All authors consent all requirements of the journal.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] W.H. Xu, J.H. Yu, A novel approach to information fusion in multi-source datasets: a granular computing viewpoint, Inf. Sci. 378 (2017) 410–423.
[2] W.H. Xu, Y.T. Guo, Generalized multigranulation double-quantitative decision-theoretic rough set, Knowl.-Based Syst. 105 (1) (2016) 190–205.
[3] K.H. Yuan, W.H. Xu, W.T. Li, W.P. Ding, An incremental learning mechanism for object classification based on progressive fuzzy three-way concept, Inf. Sci. 584 (1) (2022) 127–147.

[4] X.W. Chen, W.H. Xu, Double quantitative multigranulation rough fuzzy set based on logical operations in multi-source decision systems, Int. J. Mach. Learn. Cybern. (2021), https://doi.org/10.1007/s13042-021-01433-2.
[5] L.A. Zadeh, Fuzzy Sets and Information Granulation, Advances in Fuzzy Set Theory and Applications, North-Holland Publishing, Amsterdam, 1997.
[6] L. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.
[7] J.R. Hobbs, Granularity, in: Proceedings of the Ninth International Joint Conference on Artificial Intelligence, Los Angeles, CA, 1985.
[8] B. Zhang, L. Zhang, Theory and Applications of Problem Solving, North-Holland Publishing, Amsterdam, 1992.
[9] T.Y. Lin, Granular computing: fuzzy logic and rough sets, in: Computing with Words in Information/Intelligent Systems, Physica-Verlag, 1999.
[10] T.Y. Lin, Data mining and machine oriented modeling: a granular computing approach, J. App. Intell. 13 (2) (2000) 113–124.
[11] T.Y. Lin, Granular computing rough set perspective, Newsl. IEEE Comput. Intell. Soc. 2 (4) (2005) 1543–4281.
[12] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 111 (1998) 239–259.
[13] Y.Y. Yao, A partition model of granular computing, LNCS Trans. Rough Sets 1 (2004) 232–253.
[14] Y.Y. Yao, Information granulation and rough set approximation, Int. J. Intell. Syst. 16 (1) (2001) 87–104.
[15] D.Y. Li, H.J. Meng, X.M. Shi, Subordinate cloud and subordinate cloud generator, J. Comput. Res. Dev. 6 (1995) 15–20.
[16] Z. Zheng, H. Hu, Z.Z. Shi, Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing: Tolerance Relation Based Granular Space, Vol. 3641, Springer, Berlin/Heidelberg, 2005, pp. 682–691.
[17] J. Jarinen, Approximations and Rough Sets Based on Tolerances, Springer, Berlin/Heidelberg, 2005, pp. 182–189.
[18] M. Landowski, Shadowed numbers and their standard and multidimensional arithmetic, Inf. Sci. 507 (2020) 485–502.
[19] Y. Zhang, J.T. Yao, Game theoretical approach to shadowed sets: a three way trade off perspective, Inf. Sci. 507 (2020) 540–552.
[20] C.Z. Wang, Y. Huang, M.W. Shao, Q.H. Hu, D.G. Chen, Feature selection based on neighborhood self-information, IEEE Trans. Cybern. 50 (9) (2020) 4031–4042.
[21] C.Z. Wang, Y. Wang, M.W. Shao, Y.H. Qian, D.G. Chen, Fuzzy rough attribute reduction for categorical data, IEEE Trans. Fuzzy Syst. 28 (5) (2020) 818–830.
[22] C.Z. Wang, Y. Huang, M.W. Shao, X.D. Fan, Fuzzy rough set-based attribute reduction using distance measures, Knowl.-Based Syst. 164 (2019) 205–212.
[23] J.H. Yu, M.H. Chen, W.H. Xu, Dynamic computing rough approximations approach to time-evolving information granule interval-valued ordered information system, Appl. Soft Comput. 60 (1) (2017) 18–29.
[24] W.H. Xu, W.T. Li, Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets, IEEE Trans. Cybern. 46 (2) (2016) 366–379.
[25] W.H. Xu, K.H. Yuan, W.T. Li, Dynamic updating approximations of local generalized multigranulation neighborhood rough set, Appl. Intell. (2022), https://doi.org/10.1007/s10489-021-02861-x.
[26] W.T. Li, W.H. Xu, X.Y. Zhang, J. Zhang, Updating approximations with dynamic objects based on local multigranulation rough sets in ordered information systems, Artif. Intell. Rev. (2021), https://doi.org/10.1007/s10462-021-10053-9.
[27] Y.Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180 (2010) 341–353.
[28] Y.Y. Yao, An outline of a theory of three-way decisions, in: RSCTC 2012, in: LNCS (LNAI), vol. 7413, pp. 1–17.
[29] Y.Y. Yao, Three-way decisions and cognitive computing, Cogn. Comput. 8 (2016) 543–554.
[30] Y.Y. Yao, Three-way decision and granular computing, Int. J. Approx. Reason. 103 (2018) 107–123.
[31] J.H. Li, C.C. Huang, J.J. Qi, Y.H. Qian, W.Q. Liu, Three-way cognitive concept learning via multi-granularity, Inf. Sci. 378 (2017) 244–263.
[32] H. Sakai, M. Nakata, J. Watada, NIS-Apriori-based rule generation with three-way decisions and its application system in SQL, Inf. Sci. 507 (2020) 755–771.
[33] J.J. Qi, T. Qian, L. Wei, The connections between three-way and classical concept lattices, Knowl.-Based Syst. 91 (2016) 143–151.
[34] Y.Y. Yao, Interval sets and three-way concept analysis in incomplete contexts, Int. J. Mach. Learn. Cybern. 8 (2017) 3–20.
[35] B.H. Long, W.H. Xu, X.Y. Zhang, L. Yang, The dynamic update method of attribute-induced three-way granular concept in formal contexts, Int. J. Approx. Reason. 126 (2020) 228–248, https://doi.org/10.1016/j.ijar.2019.12.014.
[36] M.K. Afridi, N. Azam, J.T. Yao, E. Alanazi, A three-way clustering approach for handling missing data using GTRS, Int. J. Approx. Reason. 98 (2018) 11–24.
[37] P.X. Wang, Y.Y. Yao, CE3: a three-way clustering method based on mathematical morphology, Knowl.-Based Syst. 155 (2018) 54–65.
[38] H. Yu, Three-way decisions and three-way clustering, in: IJCRS 2018, in: LNCS (LNAI), vol. 11103, pp. 13–28.
[39] H. Yu, X.C. Wang, G.Y. Wang, X.H. Zeng, An active three-way clustering method via low-rank matrices for multi-view data, Inf. Sci. 507 (2020) 823–839.
[40] J.F. Xu, Y.J. Zhang, D.Q. Miao, Three-way confusion matrix for classification: a measure driven view, Inf. Sci. 507 (2020) 772–794.
[41] X.D. Yue, Y.F. Chen, D.Q. Miao, H. Fujita, Fuzzy neighborhood covering for three-way classification, Inf. Sci. 507 (2020) 795–808.
[42] Y. Fang, F. Min, Cost-sensitive approximate attribute reduction with three-way decisions, Int. J. Approx. Reason. 104 (2019) 148–165.
[43] W.W. Li, X.Y. Jia, L. Wang, B. Zhou, Multi-objective attribute reduction in three-way decision-theoretic rough set model, Int. J. Approx. Reason. 105 (2019) 327–341.
[44] X.A. Ma, Y.Y. Yao, Three-way decision perspectives on class-specific attribute reducts, Inf. Sci. 450 (2018) 227–245.
[45] Q.H. Zhang, D.Y. Xia, K.L. Liu, G.Y. Wang, A general model of decision-theoretic three-way approximations of fuzzy sets based on a heuristic algorithm, Inf. Sci. 507 (2020) 522–539.
[46] T. Feng, H.T. Fan, J.S. Mi, Uncertainty and reduction of variable precision multi-granulation fuzzy rough sets based on three-way decisions, Int. J. Approx. Reason. 85 (2017) 36–58.
[47] X.Y. Jia, Z. Deng, F. Min, D. Liu, Three-way decisions based feature fusion for Chinese irony detection, Int. J. Approx. Reason. 113 (2019) 324–335.
[48] Y.T. Yan, Z.B. Wu, X.Q. Du, J. Chen, S. Zhao, Y.P. Zhang, A three-way decision ensemble method for imbalanced data oversampling, Int. J. Approx. Reason. 107 (2019) 1–16.
[49] C. Zhang, D.Y. Li, J.Y. Liang, Multi-granularity three-way decisions with adjustable hesitant fuzzy linguistic multi-granulation decision theoretic rough sets over two universes, Inf. Sci. 507 (2020) 665–683.
[50] H. Hart, Structures of influence and cooperation-conflict, Int. Interact. 1 (1974) 141–162.
[51] Z. Pawlak, On conflicts, Int. J. Man-Mach. Stud. 21 (1984) 127–134.
[52] Z. Pawlak, An inquiry into anatomy of conflicts, Inf. Sci. 109 (1998) 65–78.
[53] G.M. Lang, D.Q. Miao, H. Fujita, Three-way group conflict analysis based on Pythagorean fuzzy set theory, IEEE Trans. Fuzzy Syst., https://doi.org/10.1109/TFUZZ.2019.2908123.
[54] B.Z. Sun, X.T. Chen, L.Y. Zhang, W.M. Ma, Three-way decision making approach to conflict analysis and resolution using probabilistic rough set over two universes, Inf. Sci. 807 (2020) 809–822.
[55] Y.Y. Yao, Three-way conflict analysis: reformulations and extensions of the Pawlak model, Knowl.-Based Syst. 180 (2019) 26–37.
[56] M.F. Sun, J.T. Chen, Y. Zhang, S.Z. Shi, A new method of feature selection for flow classification, Phys. Proc. 24 (2012) 1729–1736.
[57] L.S. Kong, X.F. Ren, Y.J. Fan, Study on assessment method for computer network security based on rough set, in: 2009 IEEE International Conference on Intelligent Computing and Intelligent Systems, Shanghai, China, 2019.
[58] Z. Pawlak, Information systems, theoretical foundations, Inf. Sci. 6 (1981) 205–218.

[59] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[60] Z. Pawlak, Approximations in the space (μ, π), Demonstr. Math. 16 (1983) 761–769.
[61] W. Zhu, Topological approaches to covering rough sets, Inf. Sci. 177 (6) (2007) 1499–1508.
[62] Q.H. Hu, D.R. Yu, J.F. Liu, C.X. Wu, Neighborhood rough set based heterogeneous feature subset selection, Inf. Sci. 178 (2008) 3577–3594.
[63] S. Udhaya kumar, H. Hannah Inbarani, A novel neighborhood rough set based classification approach for medical diagnosis, Proc. Comput. Sci. 47 (2015) 351–359.
[64] Q. Wang, Y.H. Qian, X.Y. Liang, Q. Guo, J.Y. Liang, Local neighborhood rough set, Knowl.-Based Syst. 153 (2018) 53–64.
[65] Q.Z. Kong, X.W. Zhang, W.H. Xu, S.T. Xie, Attribute reducts of multi-granulation information system, Artif. Intell. Rev. 53 (2) (2020) 1353–1371.
[66] C.Z. Wang, Y. Wang, M.W. Shao, Y.H. Qian, D.G. Chen, Fuzzy rough attribute reduction for categorical data, IEEE Trans. Fuzzy Syst. 28 (5) (2020) 818–830.
[67] X.L. Bai, Z.Q. Yun, D. Xuan, et al., Optimal patterns for four-connectivity and full coverage in wireless sensor networks, IEEE Trans. Mob. Comput. 9 (3) (2009) 435–448.